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Some Interesting Experiments: A Study of Lukasiewicz's Infinite-Valued Sentential Calculus*

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1. Winning the Game, Some Rules, Some Challenges

Just as there are many ways to win at poker—five-card draw, seven-card stud, and anaconda, for example—there are many ways to succeed at finding a sought-after proof. In this notebook, you will learn about a number of approaches that sometimes lead to victory. Strategy is the key to winning at cards, and strategy is the key to winning at proof finding. While many researchers work alone and others work with colleagues, here (as in other notebooks found on this website) you will witness my search (for various proofs) with the aid of an assistant, specifically, an automated reasoning program.

The program of my choice is William McCune's OTTER, chosen because I am so familiar with it and, far more important, because it enables me to apply diverse strategies and varied approaches. (As in the various notebooks found on this website, the approaches that are offered do not require the use of OTTER; see http://www.mcs.anl.gov/AR/otter/.) Yes, many newer reasoning programs offer greater speed, if measured in conclusions drawn per CPU-second; but, to my knowledge, none of them offers the full set of features I rely upon when seeking a proof. Those features include the ability, with max_weight, to restrict the retention of deduced conclusions based on their complexity, the ability to restrict conclusion retention based on the number of distinct variables present, and the ability to change max_weight during the run without intervention of the user. In this notebook, you will see various features being utilized, both in the discussion and in the input files I present.

To possibly add to your enjoyment and perhaps intrigue you, I include challenges, implicitly and explicitly. to attempt to meet a challenge, you need not be an expert in automated reasoning or in the area in focus. Further, neither expertise in logic nor expertise in mathematics is required for you to glean much from this notebook. Your knowledge, intuition, and conjectures can be employed in a manner that will influence OTTER's actions and, perhaps, increase the likelihood of completing the given assignment.

As you will find in this notebook, I employ various strategies and approaches to direct and restrict OTTER's reasoning. With this program, reminiscent of the remark about poker, many ways are offered by it to achieve your objective, whether the target is a shorter proof than in hand or a first proof. Of course, the power of OTTER quickly presents a huge number of conclusions to consider. Indeed, sometimes millions and millions of conclusions are drawn before the sought-after proof is returned; of these, sometimes well over one million are retained. Therefore, strategy is needed, some to restrict the reasoning, some to direct the reasoning, and some of neither class.

Very much in focus here is the *subformula strategy*, reviewed in Section 2 and discussed in detail in two other notebooks, namely, "The Subformula Strategy: Coping with Complex Expressions" and "A

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Startling Result, Some Challenges Met, but Some Still Remain: More Coping with Complex Expressions". The area of chief interest in this notebook is Lukasiewicz's infinite-valued sentential calculus (which i often denote with *MV*), both the implicational fragment and the full logic. Such areas can usually be axiomatized in various ways; indeed, often there exists a small set of formulas from which all of the theorems of the area can be deduced by applying an appropriate inference rule. The following set of clauses provides a familiar axiom system for the implicational fragment of Lukasiewicz's infinite-valued sentential calculus, where the function *i* denotes implication, the predicate *P* denotes "provability", and variables (such as *u*, *V*, *W*, and *x*) are universally quantified (meaning "for all"). (A percent sign, for OTTER, has the program treat the remainder of the line as comment.)

```
% Following 4-basis for MV implicational fragment.
P(i(x,i(y,x))). % Simp, MV1
P(i(i(x,y),i(i(y,z),i(x,z)))). % Syl, MV2
P(i(i(i(x,y),y),i(i(y,x),x))). % Inversion, MV3
P(i(i(i(x,y),i(y,x)),i(y,x))). % Linearity, MV5
```

(The numbering, *MV*1, *MV*2, *MV*3, *MV*5, will be explained shortly.)

For the curious, this infinite-valued logic is weaker than is classical, two-valued sentential calculus; indeed the given axioms, for the former, can be deduced from an axiom system for the latter. To obtain the full logic, along with his other four given axioms, Lukasiewicz also included the following axiom (in clause form, where the function *n* denotes negation).

```
P(i(i(n(x),n(y)),i(y,x))). % MV4
```

More than fifteen years ago, thanks to McCune (if memory serves), I was told of this area of logic and informed that it is sometimes called multi-valued logic. I therefore often use the letters *MV* to name the formulas, denoting as *MV*4 the fourth axiom and denoting the four that axiomatize the implicational fragment as *MV*1, *MV*2, *MV*3, and *MV*5.

I was also informed that C. A. Meredith proved that *MV*5 is dependent on the other four axioms for the full logic. The corresponding theorem might prove most challenging, with or without reliance on some computer program. If you seek a proof of the dependence of *MV*5 on *MV*1 through *MV*4 with a program that offers what is needed (in the following context), you might enjoy the challenge of seeking a proof in which the program is not permitted to retain formulas in which double negation occurs. A double-negation formula contains at least one expression of the form $n(n(t))$ for some term *t*. My introduction of double negation calls for a pause to present two anecdotes, with a small taste of a restriction strategy.

Quite a few years ago, I found that OTTER—because of the huge space of conclusions it might draw—was finding it difficult to complete some of the proofs I sought. At the time, I was studying areas of logic in which two connectives (operators) were present, namely, *i* for implication and *n* for negation. I needed a strategy that might restrict the program's search for new and crucial information dramatically, or at least reorder the search. Although perhaps counterintuitive, I decided to instruct OTTER to discard any newly deduced conclusion if the conclusion contained at least one double-negation term. As those who know of my approach and for those who might ask a most reasonable question, I had no idea whether this restriction would prevent the program from finding proofs where, in fact, proofs existed, that is, *refutation incompleteness*. As it turned out, OTTER returned the various proofs I sought and in far, far less CPU time than I expected. Therefore, at the time and still today I almost always, where appropriate, rely on the double-negation-restriction strategy, which brings me to the first small anecdote.

My colleague Branden Fitelson, now a professor at UC Berkeley, informed me that (in effect) I had not followed what seemed to be an implicit requirement in such studies: the need to rely on double negation part of the time, for some of the needed deduced steps. He asked me why the strategy seemed so effective. Now I never know how to answer a "why" question without making a wild guess, perhaps amounting to lying, in that I do not ask myself such questions. I am concerned mainly with "how" questions, how to get the program to be more effective. At the time those years ago, my answer to Fitelson was that I suspected the density of proofs was much higher when the double-negation strategy was used. Perhaps a more accurate answer asserts that the space of deducible conclusions is tremendously perturbed, moving proofs of interest to the front (so to speak). In other words, without the use of the strategy the key conclusion might, for example, be numbered 20,000,000, among the deduced clauses; whereas, with the strategy, the key might be numbered 1,000,000. Fitelson wondered whether I could always find a double-negation-free proof, except in the cases where the conclusion involves double negation, which brings me to the second anecdote.

I did have one case in which I could not do so. When in discussions with another colleague, Michael Beeson, I mentioned this case, he went off and programmed a means to produce such, if it existed. He shortly sent me a long, long proof that was free of double negation. More important, he and another colleague of ours, Robert Veroff, eventually published a paper (with me) giving conditions for when a double-negation-free proof could be found. Thus you now have in hand a small sample of how a restriction strategy can be used, how its use enlightened other researchers, and how a focus on the strategy led to a most charming result. By the way, before OTTER and I attacked the question of the dependence of *MV*5, I believe the literature's best proof consisted of 37 deduced steps. The program eventually found a 30-step proof, a proof that avoids double negation. The inference rule that was used is called *condensed detachment*, the following.

-P(i(x,y)) | -P(x) | P(y).

That rule dominates what I offer here. If you can find a proof of this dependence that relies on condensed detachment alone and whose length (measured in deduced steps) is strictly less than 30, and if you are the first to do so, you will win a prize I have offered.

With the two anecdotes concluded, next in order is a discussion of other axiom systems for the implicational fragment of Lukasiewicz's infinite-valued sentential calculus, and also for the entire logic. Quite often, a researcher seeks to find a single axiom for a field, a single formula or equation from which the theorems of the field can all be deduced, even if some of the proofs are indeed lengthy. Certainly, my colleague D. Ulrich makes such searches, many of which are successful. The Romanian logician Adrian Rezus produced a means for constructing single axiom in some cases. Indeed, for the implicational fragment of infinite-valued sentential calculus, the following is a Rezus-style single axiom (provided to me by Ken Harris).

P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
i(i(v14,i(v15,v14)),v16))),v16)).

Its length, not counting the predicate symbol, parentheses, or commas, is 69. Its variable richness (the number of distinct variables) is 17. From what I know, the literature does not offer a proof relying on condensed detachment that derives this Rezus-style axiom from a known basis, nor does the literature offer the converse. Because of the length of the Rezus-style single axiom, you might correctly expect that it would probably be difficult to complete a proof deriving it. So, naturally, I sought proofs of the corresponding two theorems (for the implicational fragment of the Lukasiewicz infinite-valued sentential calculus). The approaches I took and the results are presented later in this notebook. You might enjoy the challenge of trying to prove either theorem, unaided or with a program; for example, you might try to prove the given Rezus formula from the 4-basis (for *MV*-implicational) consisting of *MV*1, *MV*2, *MV*3, and *MV*5, and then try to derive the given 4-basis from the Rezus formula.

Especially if you are interested in logic, a natural question focuses on the possibility that a shorter single axiom exists for the implicational fragment of *MV*. Ulrich answered that question with the following single axiom; see http://web.ics.purdue.edu/~dulrich/C-pure-LNo-page.htm.

P(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7))).

This most satisfying axiom has length 37 and variable richness 8, in contrast to the Rezus axiom that has length 69 and variable richness 17. Again, a challenge exists, that of deriving the Ulrich axiom from some known basis, and conversely. Later in this notebook, I shall present the approach I took and the results.

As for the entire Lukasiewicz infinite-valued logic, Ulrich sent me the following Rezus-style single axiom.

P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)).

Not counting the predicate symbol, commas, or parentheses, the length of this single axiom is 67; its variable richness is 17. Similar to the earlier observation about *MV*-implicational, two theorems (for the entire infinite-valued logic) merit proving regarding this Rezus-style single axiom in focus. The first, and certainly the most difficult to prove, asks for a proof deriving the formula from the given 4-basis for infinite-valued sentential calculus, namely, *MV*1 through *MV*4. (By way of a reminder, *MV*5 is not in play here in that it is dependent on 1 through 4.) That theorem, deriving the 67-symbol Rezus formula, shows that the formula itself is a theorem of the logic. The second theorem asks for a proof that derives, from the Rezus-style axiom, the 4-basis in focus here, thus showing that the Rezus formula is in fact an axiom system (of course, consisting of but one member). Later in this notebook, I shall present my approaches and results for the two theorems.

I now come to a monumental challenge, one that, if met, would be most appreciated by various logicians. You have in hand a Rezus-style single axiom for the implicational fragment of the Lukasiewicz infinite-valued sentential calculus, sometimes called many-valued. You also have in hand the Ulrich 37-symbol single axiom for this area of logic. Finally, you have in hand a Rezus-style single axiom for the entire infinite-valued sentential calculus. What is missing is a “nice” single axiom for the entire calculus, one in the spirit of Ulrich. Yes, the challenge is to find one formula so powerful that it serves as a single axiom for the full Lukasiewicz infinite-valued sentential calculus, a formula that is far shorter than the 67-symbol Rezus axiom, perhaps one of length 40 or less. Of course, I do not know whether such exists.

2. Simultaneous Targets

From the viewpoint of logic, the target was to obtain a proof of the Rezus-style single axiom for the implicational fragment of the Lukasiewicz infinite-valued sentential calculus. The hypotheses to be used were *MV*1, *MV*2, *MV*3, and *MV*5, as given in Section 1; *MV*4 is not in play here in that the focus is on the implicational fragment, and not on the entire Lukasiewicz infinite-valued logic. From the viewpoint of automated reasoning, the target was to again test the subformula strategy, to be reviewed almost immediately. To test that strategy, I would need to make various choices of options and assignments to diverse parameters.

The subformula strategy was formulated to sharply increase the likelihood of finding a proof when the target is a complex expression. At least for OTTER—certainly for many researchers and, I suspect, for most reasoning programs—simpler expressions are easier to deal with than are more complex. When OTTER chooses the next item for inference-rule initiation, ordinarily the simplest available is chosen. The user can—and this is crucial to the subformula strategy—include items that will cause the program to treat complex expressions as simple. You will be presented with appropriate input files illustrating this property. Abstractly, to use the subformula strategy, you choose one or more subformulas (not necessarily proper) of the target(s) and assign to each some small value of your choosing. For OTTER, you place templates in an appropriate file, say, weight_list(pick_and_purge), that correspond to your choices. A small example illustrates how it works.

A powerful formula that occurs in various axiom systems for two-valued classical sentential calculus is termed by Lukasiewicz thesis_35, the following.

P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).

The following two weight templates each contain a subformula of thesis_35. (I note that variables in a weight template are treated as indistinguishable, just denoting that a variable occupies the corresponding position; so you culd set all variables that occur in a template to the same variable.)

weight(i(x,i(y,z)),2).
weight(i(i(x,y),i(x,z)),2).

Of course, various other expressions occur in thesis_35 that could be used in the subformula strategy, including the entire clause. The assignment of 2 to each of the given subformulas causes OTTER to consider any matching expression within a formula to be treated as if it consisted of exactly two symbols. To be a "match" just requires that all variables be ignored in the sense that all variables are considered to be indistinguishable. Therefore, rather than being treated as consisting of seven symbols, the expression i(i(u,y),i(v,y)) within a conclusion is treated as consisting of two symbols. This reduction of five at various points in a complex formula (or equation) can result in the item being treated as simple. Putting it a bit differently, long expressions, as far as a program is concerned when applying the subformula strategy, can become short.

How fortunate that I know so many people with such diverse skills. Ross Overbeek came to the rescue, giving me a program that generates the subformulas when an OTTER clause is supplied. Before that program was given to me, he supplied (with his program) a full set of subformulas when I sent him a target. Yes, I sent him the Rezus-style single axiom for the implicational fragment of the Lukasiewicz infinite-valued sentential calculus. I took what he sent and implanted it, the following, in an appropriate input file.

```
% Following 25, including target, are subformulas of the Rezus for the implicational of MV,
% from Ross.
weight(i(v,u),2).
weight(i(x,i(y,x)),2).
weight(i(w,i(v6,w)),2).
weight(i(i(v8,v9),v9),2).
weight(i(i(v9,v8),v8),2).
weight(i(i(v,z),i(v,u)),2).
weight(i(v14,i(v15,v14)),2).
weight(i(i(w,i(v6,w)),v7),2).
weight(i(i(v11,v12),i(v12,v11)),2).
weight(i(i(v14,i(v15,v14)),v16),2).
weight(i(i(z,u),i(i(v,z),i(v,u))),2).
weight(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),2).
weight(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),2).
weight(i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10),2).
weight(i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13),2).
weight(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),2).
weight(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),2).
weight(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),2).
weight(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),2).
weight(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),2).
weight(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),2).
weight(i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16)),2).
weight(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),2).
weight(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16),2).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16)),2).
```

Actually, as is obvious, I did not include the subformulas that are each themselves a variable. I chose the assignment of the value 2 to each, conjecturing (from many years of experimentation with McCune's program) that that might enable OTTER to find the desired proof that derives the Rezus-style formula from axioms 1, 2, 3, and 5. I made two experiments, in which the only difference was the assignment of the value to max_distinct_vars; in one, I chose the value 18, and in the other the value 17. The choice of 18 was motivated by the thought that, although 17 equals the number of distinct variables in the Rezus formula, perhaps a bit more latitude would be needed. Of course, as indicated, I used just the given subformulas to guide the search.

More is needed with OTTER, unless you wish to rely on its autonomous mode, set(auto_mode). I seldom choose this approach, instead deciding on which options to use and which values to assign to the various parameters. Therefore, even though my two experiments were designed to both test the subformula strategy and seek a first-ever condensed-detachment proof of the Rezus formula from the hypotheses consisting of axioms 1, 2, 3, and 5, other decisions had to be made. At the top level, I had to choose between a level-saturation approach and a complexity-preference approach. In the former, the program begins with the input clauses in list(sos), those of level 0, and first deduces (within whatever constraints are placed on the program) the level-1 conclusions, the children of the level-0 clauses in list(sos). The input clauses not in list(sos) are used to complete applications of, in this case, condensed detachment. For the case under study, the only important occupant of list(usable), outside of the initial list(sos), is a clause that (together with hyperresolution) has the program apply condensed detachment, a clause given near the end of Section 1. In contrast, with a complexity-preference search, the program repeatedly chooses the simplest item not yet used and relies on it to initiate inference-rule application. Ordinarily, complexity is measured in terms of symbol count; the measure is modified, depending on what items are found, for example, in weight_list(pick_and_purge). (Again, I note that you are not required to rely on OTTER; you can instead rely on another reasoning program if it offers what you need, or you can program what you wish to use, taken from one of these notebooks.)

I chose what might be termed an intermediate course between the two given approaches. In particular, I did not include set(sos_queue), which has OTTER conduct a level-saturation approach. However, I did include the command set(input_sos_first), which instructs the program to first focus for inference-rule initiation on the items in the initial set of support, and I did include assign(pick_given_ratio,1). That last inclusion has the program choose, for directing the reasoning with the chosen inference rule(s), 1 clause by complexity preference, 1 by first come first serve, then 1 by complexity preference, 1 by first come first serve, 1, 1, and the like. In other words, so-to-speak, I mixed the two major approaches.

Overbeek likes the approach of starting with a small assigned value to max_weight and gradually increasing the assigned value until success occurs, if it does. The max_weight parameter places an upper limit on the complexity of newly retained information. Too small a value ordinarily causes the program to run out of conclusions to draw, signified with sos empty. What worked was an assignment of the value 8 to max_weight; if memory serves, smaller values resulted in running out of conclusions to draw before the assignment was complete. The included command set(order_history) has the program list the ancestors of a deduced item (with hyperresolution) nucleus (here for condensed detachment), major premiss, minor premiss. The inclusion of assign(max_proofs,-1) has the program complete as many proofs as it can within the memory and time constraints chosen by the user. The following input file illustrates that which has just been discussed.

An Input File for Proving Rezus with Subformulas

```
set(hyper_res).
assign(max_mem,880000).
% assign(max_seconds,3).
% set(sos_queue).
set(input_sos_first).
assign(max_weight,8).
% assign(change_limit_after,200).
% assign(new_max_weight,8).
assign(max_proofs,-1).
```

```
assign(max_distinct_vars,17).
assign(pick_given_ratio,1).
assign(bsub_hint_wt,1).
assign(report,5400).
set(keep_hint_subsumers).
% set(keep_hint_equivalents).
% set(ancestor_subsume).
set(back_sub).
set(order_history).
% set(process_input).
clear(print_kept).

weight_list(pick_and_purge).
% Following 25, including target, are subformulas of the Rezus for the implicational of MV,
% from Ross.
weight(i(v,u),2).
weight(i(x,i(y,x)),2).
weight(i(w,i(v6,w)),2).
weight(i(i(v8,v9),v9),2).
weight(i(i(v9,v8),v8),2).
weight(i(i(v,z),i(v,u)),2).
weight(i(v14,i(v15,v14)),2).
weight(i(i(w,i(v6,w)),v7),2).
weight(i(i(v11,v12),i(v12,v11)),2).
weight(i(i(v14,i(v15,v14)),v16),2).
weight(i(i(z,u),i(i(v,z),i(v,u))),2).
weight(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),2).
weight(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),2).
weight(i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10),2).
weight(i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13),2).
weight(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),2).
weight(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),2).
weight(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),2).
weight(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),2).
weight(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),2).
weight(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),2).
weight(i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16)),2).
weight(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),2).
weight(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
  i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16),2).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16)),2).  % Rezus-style single axiom for MV implicational
end_of_list.
```

```
list(usable).
-P(i(x,y)) | -P(x) | P(y).
%  -P(i(a,a)) | -P(i(i(a,i(b,c)),i(b,i(a,c)))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(a,i(a,b)),i(a,b))) |
   $ANS(all).
end_of_list.

list(sos).
%  Following 4-basis, used by Veroff, for MV implicational fragment.
P(i(x,i(y,x))).
P(i(i(x,y),i(i(y,z),i(x,z)))).
P(i(i(i(x,y),y),i(i(y,x),x))).
P(i(i(i(x,y),i(y,x)),i(y,x))).
end_of_list.

list(passive).
%  Following 21 negs of a 21-step proof, shortest as of 05-23-08, of Rezus
%  for MV implicational, temp.ulrich.Rezus.mv.out1m3.
-P(i(a1,i(a2,i(a3,a2)))) | $ANS(REZPRF1).
-P(i(i(i(i(a1,a2),i(a3,a2)),a4),i(i(a3,a1),a4))) | $ANS(REZPRF1).
-P(i(i(i(a1,a2),a3),i(a2,a3))) | $ANS(REZPRF1).
-P(i(i(i(a1,i(a2,a1)),a3),a3)) | $ANS(REZPRF1).
-P(i(a1,i(i(a1,a2),a2))) | $ANS(REZPRF1).
-P(i(i(a1,a2),i(i(i(a3,i(a4,a3)),a1),a2))) | $ANS(REZPRF1).
-P(i(i(i(i(a1,a2),a2),a3),i(a1,a3))) | $ANS(REZPRF1).
-P(i(i(a1,a2),i(i(a3,a1),i(a3,a2)))) | $ANS(REZPRF1).
-P(i(a1,i(i(a2,i(a1,a3)),i(a2,a3)))) | $ANS(REZPRF1).
-P(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),a4),a4)) | $ANS(REZPRF1).
-P(i(i(a1,i(i(i(i(a2,a3),i(a3,a2)),i(a3,a2)),a4)),i(a1,a4))) | $ANS(REZPRF1).
-P(i(i(a1,i(i(i(i(a2,a3),a3),i(i(a3,a2),a2)),a4)),i(a1,a4))) | $ANS(REZPRF1).
-P(i(i(a1,i(i(a2,i(a3,a2)),a4)),i(a1,a4))) | $ANS(REZPRF1).
-P(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6)) | $ANS(REZPRF1).
-P(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),b6),b6)) | $ANS(REZPRF1).
-P(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
   i(i(b7,b6),b6)),b8)),b8)) | $ANS(REZPRF1).
-P(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
   i(i(b7,b6),b6)),b8)),b8),b9),b9)) | $ANS(REZPRF1).
-P(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
   i(i(b7,b6),b6)),b8)),b8),i(i(i(b9,b10),i(b10,b9)),i(b10,b9)),b11)),b11)) | $ANS(REZPRF1).
-P(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
   i(i(b7,b6),b6)),b8)),b8),i(i(i(b9,b10),i(b10,b9)),i(b10,b9)),b11)),b11),b12),b12)) | $ANS(REZPRF1).
-P(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
   i(i(b7,b6),b6)),b8)),b8),i(i(i(b9,b10),i(b10,b9)),i(b10,b9)),b11)),b11),i(i(b12,i(b13,b12)),b14)),b14)) |
   $ANS(REZPRF1).
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
   i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
   i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZPROOF1).
% Following 32, negs, prove the Rezus single for MV implicational, temp.ulrich.rezus.mv.out1m2.
-P(i(a1,i(a2,i(a3,a2)))) | $ANS(REZPRF).
-P(i(i(i(i(a1,a2),i(a3,a2)),a4),i(i(a3,a1),a4))) | $ANS(REZPRF).
-P(i(i(i(a1,a2),a3),i(a2,a3))) | $ANS(REZPRF).
-P(i(a1,i(i(i(a2,a3),a3),i(i(a3,a2),a2)))) | $ANS(REZPRF).
-P(i(a1,i(i(i(a2,a3),i(a3,a2)),i(a3,a2)))) | $ANS(REZPRF).
-P(i(i(i(a1,i(a2,a1)),a3),a3)) | $ANS(REZPRF).
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-P(i(i(a1,a2),i(i(i(a1,a3),a4),i(i(a2,a3),a4)))) | $ANS(REZPRF).
-P(i(a1,i(i(a1,a2),a2))) | $ANS(REZPRF).
-P(i(i(i(i(i(a1,a2),a2),i(i(a2,a1),a1)),a3),a3)) | $ANS(REZPRF).
-P(i(i(i(i(i(a1,a2),i(a2,a1)),i(a2,a1)),a3),a3)) | $ANS(REZPRF).
-P(i(a1,a1)) | $ANS(REZPRF).
-P(i(i(a1,a2),i(i(i(a3,i(a4,a3)),a1),a2))) | $ANS(REZPRF).
-P(i(i(i(i(i(a1,a2),a3),i(i(a4,a2),a3)),b),i(i(a1,a4),b))) | $ANS(REZPRF).
-P(i(i(i(i(a1,a2),a2),a3),i(a1,a3))) | $ANS(REZPRF).
-P(i(i(i(a1,a1),a2),a2)) | $ANS(REZPRF).
-P(i(i(a1,a2),i(i(a3,a1),i(i(a2,a4),i(a3,a4))))) | $ANS(REZPRF).
-P(i(i(a1,a2),i(i(a3,a1),i(a3,a2)))) | $ANS(REZPRF).
-P(i(i(a1,a2),i(i(i(a3,a3),a1),a2))) | $ANS(REZPRF).
-P(i(i(a1,i(i(i(i(a2,a3),a3),i(i(a3,a2),a2)),a4)),i(i(a4,b),i(a1,b)))) | $ANS(REZPRF).
-P(i(i(a1,i(i(a2,i(a3,a2)),a4)),i(i(a4,b),i(a1,b)))) | $ANS(REZPRF).
-P(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),a4),a4)) | $ANS(REZPRF).
-P(i(i(a1,i(i(i(i(a2,a3),i(a3,a2)),i(a3,a2)),a4)),i(a1,a4))) | $ANS(REZPRF).
-P(i(i(a1,i(i(a2,i(a3,a2)),a4)),i(a1,a4))) | $ANS(REZPRF).
-P(i(i(i(a1,a1),a2),i(i(a2,a3),a3))) | $ANS(REZPRF).
-P(i(i(a1,a2),i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),a1)),a2))) | $ANS(REZPRF).
-P(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),b6),b6)) | $ANS(REZPRF).
-P(i(i(a1,a2),i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),i(i(i(i(b8,b9),b9),
  i(i(b9,b8),b8)),a1)),a2))) | $ANS(REZPRF).
-P(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
  i(i(b7,b6),b6)),b8)),b8),b9),b9)) | $ANS(REZPRF).
-P(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
  i(i(b7,b6),b6)),b8)),b8),i(i(i(b9,b10),i(b10,b9)),i(b10,b9)),b11)),b11)) | $ANS(REZPRF).
-P(i(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
  i(i(b7,b6),b6)),b8)),b8),i(i(i(b9,b10),i(b10,b9)),i(b10,b9)),b11)),b11),b12),b12)) | $ANS(REZPRF).
-P(i(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
  i(i(b7,b6),b6)),b8)),b8),i(i(i(b9,b10),i(b10,b9)),i(b10,b9)),b11)),b11),i(i(b12,i(b13,b12)),b14)),b14)) |
  $ANS(REZPRF).
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZPROOF).
% Following 25 are negs of subformulas of the Rezus for MV implicational,
% treated as if they might be theorems.
-P(i(b,a4)) | $ANS(REZMV).
-P(i(a1,i(a2,a1))) | $ANS(REZMV).
-P(i(a6,i(b6,a6))) | $ANS(REZMV).
-P(i(i(b8,b9),b9)) | $ANS(REZMV).
-P(i(i(b9,b8),b8)) | $ANS(REZMV).
-P(i(i(b,a3),i(b,a4))) | $ANS(REZMV).
-P(i(b14,i(b15,b14))) | $ANS(REZMV).
-P(i(i(a6,i(b6,a6)),b7)) | $ANS(REZMV).
-P(i(i(b11,b12),i(b12,b11))) | $ANS(REZMV).
-P(i(i(b14,i(b15,b14)),b16)) | $ANS(REZMV).
-P(i(i(a3,a4),i(i(b,a3),i(b,a4)))) | $ANS(REZMV).
-P(i(i(i(b8,b9),b9),i(i(b9,b8),b8))) | $ANS(REZMV).
-P(i(i(i(b11,b12),i(b12,b11)),i(b12,b11))) | $ANS(REZMV).
-P(i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)) | $ANS(REZMV).
-P(i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)) | $ANS(REZMV).
-P(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7))) | $ANS(REZMV).
-P(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7)) | $ANS(REZMV).
```

```
-P(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),i(i(i(i(b8,b9),b9),
  i(i(b9,b8),b8)),b10))) | $ANS(REZMV).
-P(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),i(i(i(i(b8,b9),b9),
  i(i(b9,b8),b8)),b10)),b10)) | $ANS(REZMV).
-P(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),i(i(i(i(b8,b9),b9),
  i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13))) | $ANS(REZMV).
-P(i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),i(i(i(i(b8,b9),b9),
  i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13)) | $ANS(REZMV).
-P(i(i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),i(i(i(i(b8,b9),b9),
  i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))) | $ANS(REZMV).
-P(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16)))) | $ANS(REZMV).
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZMV).
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZMV).
%  Following is neg of Ulrich's 37-symbol single axiom for the implicational fragment of MV.
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7)),
  i(i(b6,b),b7))) | $ANS(UL37).
%  Following is neg of the rezus-style single axiom for the implicational fragment of MV.
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZM).
end_of_list.

list(demodulators).
%  (i(i(x,x),y) = junk).
%  (i(y,i(x,x)) = junk).
(i(x,junk) = junk).
(i(junk,x) = junk).
(P(junk) = $T).
end_of_list.
```

By way of clarification, none of the items in list(passive) play a role in OTTER's reasoning. Many of them are present merely to monitor progress; some are there to detect the completion of the sought-after proof(s), when "unit conflict" has been detected. This input file, as opposed to its almost identical copy that has an assignment of the value 18 to max_distinct_vars, found the desired proof, one of length 73 and level 15, the following.

A Proof of Rezus Based on the Sole Use of Subformulas

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on octopus.mcs.anl.gov,
Sat May 24 09:16:53 2008
The command was "otter". The process ID is 18935.

----> UNIT CONFLICT at 38158.20 sec ----> 160376 [binary,160373.1,58.1] $ANS(REZPROOF).

Length of proof is 73. Level of proof is 15.
```

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(x,i(y,x))).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(i(x,y),y),i(i(y,x),x))).
5 [] P(i(i(i(x,y),i(y,x)),i(y,x))).
58 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZPROOF).
90 [hyper,1,2,2] P(i(x,i(y,i(z,y)))).
93 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
96 [hyper,1,2,3] P(i(x,i(i(y,z),i(i(z,u),i(y,u))))).
97 [hyper,1,3,2] P(i(i(i(x,y),z),i(y,z))).
100 [hyper,1,2,4] P(i(x,i(i(i(y,z),z),i(i(z,y),y)))).
103 [hyper,1,2,5] P(i(x,i(i(i(y,z),i(z,y)),i(z,y)))).
105 [hyper,1,4,90] P(i(i(i(x,i(y,x)),z),z)).
110 [hyper,1,4,96] P(i(i(i(i(x,y),i(i(y,z),i(x,z))),u),u)).
113 [hyper,1,93,5] P(i(i(x,x),i(x,x))).
114 [hyper,1,93,3] P(i(i(x,y),i(i(i(x,z),u),i(i(y,z),u)))).
117 [hyper,1,5,113] P(i(x,x)).
120 [hyper,1,3,97] P(i(i(i(x,y),z),i(i(i(u,x),y),z))).
122 [hyper,1,97,4] P(i(x,i(i(x,y),y))).
127 [hyper,1,2,117] P(i(x,i(y,y))).
128 [hyper,1,4,100] P(i(i(i(i(i(x,y),y),i(i(y,x),x)),z),z)).
132 [hyper,1,4,127] P(i(i(i(x,x),y),y)).
139 [hyper,1,4,103] P(i(i(i(i(i(x,y),i(y,x)),i(y,x)),z),z)).
143 [hyper,1,93,105] P(i(i(x,y),i(x,i(z,y)))).
144 [hyper,1,3,105] P(i(i(x,y),i(i(i(z,i(u,z)),x),y))).
150 [hyper,1,3,122] P(i(i(i(i(x,y),y),z),i(x,z))).
154 [hyper,1,122,127] P(i(i(i(x,i(y,y)),z),z)).
168 [hyper,1,93,110] P(i(i(x,i(y,z)),i(x,i(i(z,u),i(y,u))))).
174 [hyper,1,3,132] P(i(i(x,y),i(i(i(z,z),x),y))).
178 [hyper,1,97,114] P(i(x,i(i(i(y,z),u),i(i(x,z),u)))).
179 [hyper,1,3,114] P(i(i(i(i(i(x,y),z),i(i(u,y),z)),v),i(i(x,u),v))).
186 [hyper,1,93,139] P(i(i(x,i(i(y,z),i(z,y))),i(x,i(z,y)))).
198 [hyper,1,3,154] P(i(i(x,y),i(i(i(z,i(u,u)),x),y))).
213 [hyper,1,3,178] P(i(i(i(i(i(x,y),z),i(i(u,y),z)),v),i(u,v))).
265 [hyper,1,179,110] P(i(i(x,y),i(i(y,z),i(i(z,u),i(x,u))))).
272 [hyper,1,179,150] P(i(i(x,y),i(x,i(i(y,z),z)))).
275 [hyper,1,93,150] P(i(i(x,i(y,z)),i(y,i(x,z)))).
278 [hyper,1,150,93] P(i(i(x,y),i(i(z,x),i(z,y)))).
282 [hyper,1,278,278] P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))).
288 [hyper,1,122,278] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)).
293 [hyper,1,3,278] P(i(i(i(i(x,y),i(x,z)),u),i(i(y,z),u))).
299 [hyper,1,278,154] P(i(i(x,i(i(y,i(z,z)),u)),i(x,u))).
302 [hyper,1,278,139] P(i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),i(x,u))).
309 [hyper,1,278,128] P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))).
315 [hyper,1,278,105] P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))).
340 [hyper,1,150,213] P(i(i(i(x,y),z),i(u,i(i(u,y),z)))).
394 [hyper,1,168,278] P(i(i(x,y),i(i(i(z,y),u),i(i(z,x),u)))).
544 [hyper,1,174,265] P(i(i(i(x,x),i(y,z)),i(i(z,u),i(i(u,v),i(y,v))))).
572 [hyper,1,186,275] P(i(i(x,i(i(y,x),y)),i(x,y))).

576 [hyper,1,120,275] P(i(i(i(x,y),i(z,u)),i(z,i(y,u)))).
578 [hyper,1,3,275] P(i(i(i(x,i(y,z)),u),i(i(y,i(x,z)),u))).
641 [hyper,1,3,394] P(i(i(i(i(i(x,y),z),i(i(x,u),z)),v),i(i(u,y),v))).
997 [hyper,1,186,293] P(i(i(i(i(x,y),i(x,z)),i(z,y)),i(z,y))).
1049 [hyper,1,299,288] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,v)),w)),w)).
1056 [hyper,1,272,1049] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,v)),w)),i(i(w,v6),v6))).
1169 [hyper,1,174,576] P(i(i(i(x,x),i(i(y,z),i(u,v))),i(u,i(z,v)))).
1224 [hyper,1,578,282] P(i(i(x,i(y,z)),i(y,i(i(u,x),i(u,z))))).
1262 [hyper,1,315,288] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
1265 [hyper,1,315,110] P(i(i(i(i(x,y),i(i(y,z),i(x,z))),i(i(u,i(v,u)),w)),w)).
1285 [hyper,1,122,1262] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)).
1312 [hyper,1,122,1265] P(i(i(i(i(i(i(x,y),i(i(y,z),i(x,z))),i(i(u,i(v,u)),w)),w),v6),v6)).
1453 [hyper,1,143,641] P(i(i(i(i(i(x,y),z),i(i(x,u),z)),v),i(w,i(i(u,y),v)))).
1948 [hyper,1,275,997] P(i(x,i(i(i(i(y,z),i(y,x)),i(x,z)),z))).
1964 [hyper,1,198,1948] P(i(i(i(x,i(y,y)),z),i(i(i(i(u,v),i(u,z)),i(z,v)),v))).
2193 [hyper,1,1056,544] P(i(i(i(i(i(x,x),y),i(i(z,u),y)),v),v)).
2199 [hyper,1,578,2193] P(i(i(x,i(i(i(i(y,y),z),i(i(u,v),z)),w)),i(x,w))).
2377 [hyper,1,578,1169] P(i(i(i(x,y),i(i(z,z),i(u,v))),i(u,i(y,v)))).
3504 [hyper,1,340,1964] P(i(x,i(i(x,y),i(i(i(i(z,u),i(z,y)),i(y,u)),u)))).
4311 [hyper,1,2377,572] P(i(x,i(i(i(i(x,y),i(z,z)),i(x,y)),y))).
5769 [hyper,1,1224,4311] P(i(i(i(i(x,y),i(z,z)),i(x,y)),i(i(u,x),i(u,y)))).
6152 [hyper,1,309,1285] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8)).
6229 [hyper,1,122,6152] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8),v9),v9)).
10693 [hyper,1,2199,1312] P(i(i(i(i(i(i(x,y),i(i(y,z),i(x,z))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v6),v7),
i(i(v8,v9),v7)),v10)),v10)).
10714 [hyper,1,1453,10693] P(i(x,i(i(i(i(y,z),i(i(u,v),i(u,v))),i(i(w,i(v6,w)),v7)),v7))).
10950 [hyper,1,5769,10714] P(i(i(x,i(i(i(y,z),i(i(u,v),i(u,v))),i(i(w,i(v6,w)),v7))),i(x,v7))).
32965 [hyper,1,302,6229] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)).
33028 [hyper,1,3504,32965] P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),v12),i(i(i(i(v13,v14),
i(v13,v12)),i(v12,v14)),v14))).
160330 [hyper,1,10950,33028] P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),
i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),
i(i(v12,i(v13,v12)),v14)),v14)).
160373 [hyper,1,144,160330] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
i(i(v14,i(v15,v14)),v16))),v16)).

When the companion input file was used, with an assignment of 18 to max_distinct_vars, the same exact proof was found, but twice as much CPU time was required. You might, as I have, conclude that the extra latitude was unexpectedly costly in computer time.

You might rather quickly, as my colleague Michael Beeson wondered, about the use of subformulas from the hypothesis as well as those from the target(s). Later in this notebook, if all goes as planned, I give experiments that show that it can indeed be profitable to include subformulas from both target and hypothesis.

From what I know, the given proof is the first of its kind in this context; indeed, no formal proof deriving the Rezus-style axiom from the given 4-basis existed before OTTER entered the picture. I prefer formal proofs, proofs in which one or more specific inference rules are employed whose use leads to the deduction of one item after another until the desired goal is reached. You thus have one of my reasons for enjoying the use of condensed detachment. To be precise, you might wish to have the converse, a proof showing that the Rezus formula is an axiom (for the implicational fragment), that its use can lead to the derivation of a known basis for that area of logic. I set out to obtain that proof with the following input file, again depending for advice solely on the subformula strategy.

An Input File Deriving the 4-Basis from Rezus

```
set(hyper_res).
assign(max_mem,880000).
% assign(max_seconds,3).
% set(sos_queue).
set(input_sos_first).
assign(max_weight,6).
% assign(change_limit_after,50).
% assign(new_max_weight,6).
assign(max_proofs,-1).
% assign(max_distinct_vars,19).
assign(pick_given_ratio,4).
assign(bsub_hint_wt,1).
assign(report,5400).
set(keep_hint_subsumers).
% set(keep_hint_equivalents).
set(ancestor_subsume).
set(back_sub).
set(order_history).
% set(process_input).
clear(print_kept).

weight_list(pick_and_purge).
% Following 25, including target, are subformulas of the Rezus for the implicational of MV,
% from Ross.
weight(i(v,u),1).
weight(i(x,i(y,x)),1).
weight(i(w,i(v6,w)),1).
weight(i(i(v8,v9),v9),1).
weight(i(i(v9,v8),v8),1).
weight(i(i(v,z),i(v,u)),1).
weight(i(v14,i(v15,v14)),1).
weight(i(i(w,i(v6,w)),v7),1).
weight(i(i(v11,v12),i(v12,v11)),1).
weight(i(i(v14,i(v15,v14)),v16),1).
weight(i(i(z,u),i(i(v,z),i(v,u))),1).
weight(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),1).
weight(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),1).
weight(i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10),1).
weight(i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13),1).
weight(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),1).
weight(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),1).
weight(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),1).
weight(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),1).
weight(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),
  i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),1).
weight(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
```

```
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),1).
weight(i(i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16)),1).
weight(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),1).
weight(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
  i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16),1).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16)),1).  % Rezus-style axiom for implicational MV
end_of_list.

list(usable).
-P(i(x,y)) | -P(x) | P(y).
-P(i(a,i(b,a))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(i(a,b),b),i(i(b,a),a))) | -P(i(i(i(a,b),i(b,a)),i(b,a))) |
  $ANS(MVBASISALL).
end_of_list.

list(sos).
P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)).
% Rezus-style axiom for MV
end_of_list.

list(passive).
% Following 4-basis, used by Veroff, for MV implicational fragment.
-P(i(a,i(b,a))) | $ANS(MVBASIS).
-P(i(i(a,b),i(i(b,c),i(a,c)))) | $ANS(MVBASIS).
-P(i(i(i(a,b),b),i(i(b,a),a))) | $ANS(MVBASIS).
-P(i(i(i(a,b),i(b,a)),i(b,a))) | $ANS(MVBASIS).
% Following is neg of Ulrich's 37-symbol single axiom for the implicational fragment of MV.
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7)),
  i(i(b6,b),b7))) | $ANS(UL37).
% Following is neg of the rezus-style single axiom for the implicational fragment of MV.
-(P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16))) | $ANS(REZUL).
end_of_list.

list(demodulators).
% (i(i(x,x),y) = junk).
% (i(y,i(x,x)) = junk).
(i(x,junk) = junk).
(i(junk,x) = junk).
(P(junk) = $T).
end_of_list.
```

I also used a similar input file that differed in that it did not use set(ancestor_subsume). The command set(ancestor_subsume) has the program seek in an automated manner shorter proofs if it can find such. Ancestor subsumption, for which we can thank McCune, proceeds by comparing derivation lengths to copies of the same deduced conclusions, preferring the strictly shorter. I assigned the value 6 to max_weight in part because the targets, four of them (the 4-basis) would almost be certainly easer to reach when compared with the Rezus formula as target. I assigned the value 4 to the pick_given_ratio, which has OTTER choose 4 clauses by complexity preference (for inference-rule initiation), 1 by first come first serve, 4, 1, and the like. Without the use of ancestor subsumption, OTTER presented me with a 19-step proof; with its use, the following 18-step proof was found.

An 18-step Proof That Derives from Rezus the 4-Basis

----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on lemma.mcs.anl.gov,
Wed May 21 16:42:21 2008
The command was "otter". The process ID is 3054.

-----> EMPTY CLAUSE at 23.11 sec ----> 5355 [hyper,2,15,5353,53,45] $ANS(MVBASISALL).

Length of proof is 18. Level of proof is 10.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] -P(i(a,i(b,a))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(i(a,b),b),i(i(b,a),a))) | -P(i(i(i(a,b),i(b,a)),i(b,a))) | $ANS(MVBASISALL).
3 [] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7), i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13), i(i(v14,i(v15,v14)),v16))),v16)).
14 [hyper,1,3,3] P(i(x,i(y,i(z,y)))).
15 [hyper,1,14,14] P(i(x,i(y,x))).
17 [hyper,1,3,14] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)).
21 [hyper,1,17,15] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8)).
22 [hyper,1,17,14] P(i(x,i(i(i(y,z),i(z,y)),i(z,y)))).
24 [hyper,1,21,15] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
25 [hyper,1,21,14] P(i(x,i(i(i(y,z),z),i(i(z,y),y)))).
30 [hyper,1,24,15] P(i(i(x,y),i(i(z,x),i(z,y)))).
31 [hyper,1,30,30] P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))).
35 [hyper,1,30,24] P(i(i(x,i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6))),i(x,v6))).
45 [hyper,1,22,22] P(i(i(i(x,y),i(y,x)),i(y,x))).
53 [hyper,1,25,25] P(i(i(i(x,y),y),i(i(y,x),x))).
59 [hyper,1,53,25] P(i(i(i(i(i(x,y),y),i(i(y,x),x)),z),z)).
71 [hyper,1,24,31] P(i(i(x,i(y,z)),i(x,i(y,i(u,z))))).
108 [hyper,1,24,59] P(i(x,i(i(x,y),y))).
206 [hyper,1,31,108] P(i(x,i(i(y,i(x,z)),i(y,z)))).
1003 [hyper,1,71,206] P(i(x,i(i(y,i(x,z)),i(u,i(y,z))))).
5353 [hyper,1,35,1003] P(i(i(x,y),i(i(y,z),i(x,z)))).

Next in order is another single axiom for the implicational fragment of the Lukasiewicz infinite-valued sentential calculus, specifically, the following Ulrich 37-symbol single axiom, given near the end of Section 1.

P(i(i(i(x,i(y,x)),i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7))).

You might enjoy the challenge of seeking shorter proofs for the two given theorems, perhaps markedly shorter. I plan to include in Section 6 of this notebook the results of my attempts.

3. The Ulrich 37-Symbol Single Axiom

With the preceding success in mind, I turned to the study of the Ulrich 37-symbol single axiom (the following) for the implicational fragment of *MV*.

P(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7))).

As with Rezus, my plan was to rely solely on subformulas of the Ulrich axiom to direct the search. Whereas Rezus in effect offered 25 subformulas, Ulrich offers 13, the following as they occur in an input file I shall give.

```
% Following 13 are subformulas of the Ulrich 37 for MV, from Overbeek.
weight(i(u,z),2).
weight(i(i(z,u),u),2).
weight(i(x,i(y,x)),2).
weight(i(i(v6,v),v7),2).
weight(i(i(z,u),i(u,z)),2).
weight(i(i(i(z,u),i(u,z)),z),2).
weight(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),2).
weight(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),2).
weight(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),2).
weight(i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7),2).
weight(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),2).
weight(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7)),2).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7))),2).
% Ulrich 37-symbol for MV-imp
```

I chose to rely on a level-saturation approach for reasons that currently escape me. When a level-saturation approach is relied on, the pick_given_ratio is irrelevant.

After a few experiments, I chose an assignment of the value 10 to max_weight; smaller values did not suffice. Such an assignment might indeed be unexpected in that the value 8 sufficed for the study of Rezus, a far, far more complicated formula. (I note that, if a choice of a value for max_weight is too small to complete the given assignment, the typical message is sos empty, but such is not always the case.) Two additional commands in the cont.ext of max_weight benefit from discussion.

```
assign(change_limit_after,200).
assign(new_max_weight,8).
```

The second of the two, as its syntax suggests, has the program rely on a new value for max_weight. The first of the two informs the program when to change to the new value; in the case here, after 200 clauses have been chosen to initiate inference-rule application, the new assigned value comes into play.

I assigned the value 8 to max_distinct_vars because the Ulrich single axiom relies on eight distinct variables. In that the Rezus-style single axiom in focus earlier has a variable richness of 17, again at this point the need to begin with an assignment of the value 10 to max_weight sees odd. Perhaps my version of history is a bit off; I leave it to you, if you are curious, to make the appropriate experiments.

More of history escapes me, but I think it not crucial. In particular, I chose to rely on ancestor subsumption, for reasons that are not clear. Perhaps those Greek gods were not mythological. Specifically, with the following input file—because of the use of ancestor subsumption—the program returned to me proofs of lengths 19, 16, 14, and 13.

An Input File for Deriving the Ulrich 37-Symbol Single Axiom

```
set(hyper_res).
assign(max_mem,880000).
% assign(max_seconds,3).
set(sos_queue).
```

```
set(input_sos_first).
assign(max_weight,10).
assign(change_limit_after,200).
assign(new_max_weight,8).
assign(max_proofs,-1).
assign(max_distinct_vars,8).
% assign(pick_given_ratio,1).
assign(bsub_hint_wt,1).
assign(report,5400).
set(keep_hint_subsumers).
% set(keep_hint_equivalents).
set(ancestor_subsume).
set(back_sub).
set(order_history).
% set(process_input).
clear(print_kept).

weight_list(pick_and_purge).
% Following 13 are subformulas of the Ulrich 37 for MV, from Ross.
weight(i(u,z),2).
weight(i(i(z,u),u),2).
weight(i(x,i(y,x)),2).
weight(i(i(v6,v),v7),2).
weight(i(i(z,u),i(u,z)),2).
weight(i(i(i(z,u),i(u,z)),z),2).
weight(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),2).
weight(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),2).
weight(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),2).
weight(i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7),2).
weight(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),2).
weight(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7)),2).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7))),2).
% Ulrich 37-symbol for MV-imp
end_of_list.

list(usable).
-P(i(x,y)) | -P(x) | P(y).
% -P(i(a,a)) | -P(i(i(a,i(b,c)),i(b,i(a,c)))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(a,i(a,b)),i(a,b))) |
  $ANS(all).
end_of_list.

list(sos).
% Following 4-basis, used by Veroff, for MV implicational fragment.
P(i(x,i(y,x))).
P(i(i(x,y),i(i(y,z),i(x,z)))).
P(i(i(i(x,y),y),i(i(y,x),x))).
P(i(i(i(x,y),i(y,x)),i(y,x))).
end_of_list.

list(passive).
% Following 13 are negs of subformulas, including the Ulrich axiom, of his axiom.
-P(i(a4,a3)) | $ANS(UL).
-P(i(i(a3,a4),a4)) | $ANS(UL).
```

```
-P(i(a1,i(a2,a1))) | $ANS(UL).
-P(i(i(b6,b),b7)) | $ANS(UL).
-P(i(i(a3,a4),i(a4,a3))) | $ANS(UL).
-P(i(i(i(a3,a4),i(a4,a3)),a3)) | $ANS(UL).
-P(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3))) | $ANS(UL).
-P(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6))) | $ANS(UL).
-P(i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6))) | $ANS(UL).
-P(i(i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7)) | $ANS(UL).
-P(i(i(a1,i(a2,a1)),i(i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7))) | $ANS(UL).
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7)),i(i(b6,b),b7))) |
  $ANS(UL).
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7)),i(i(b6,b),b7))) |
  $ANS(ULAX).
end_of_list.

list(demodulators).
%  (i(i(x,x),y) = junk).
%  (i(y,i(x,x)) = junk).
(i(x,junk) = junk).
(i(junk,x) = junk).
(P(junk) = $T).
end_of_list.
```

A 13-Step Proof Deriving the Ulrich 37-Symbol Single Axiom from the 4-Basis

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on jaguar.mcs.anl.gov,
Wed May 28 13:30:06 2008
The command was "otter".  The process ID is 25649.

----> UNIT CONFLICT at 141.54 sec ----> 10583 [binary,10582.1,18.1] $ANS(ULAX).

Length of proof is 13.  Level of proof is 7.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(x,i(y,x))).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(i(x,y),y),i(i(y,x),x))).
5 [] P(i(i(i(x,y),i(y,x)),i(y,x))).
18 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7)),
  i(i(b6,b),b7))) | $ANS(ULAX).
22 [hyper,1,2,2] P(i(x,i(y,i(z,y)))).
23 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
26 [hyper,1,3,4] P(i(i(i(i(x,y),y),z),i(i(i(y,x),x),z))).
28 [hyper,1,3,5] P(i(i(i(x,y),z),i(i(i(y,x),i(x,y)),z))).
30 [hyper,1,4,22] P(i(i(i(x,i(y,x)),z),z)).
50 [hyper,1,26,3] P(i(i(i(x,y),y),i(i(x,z),i(i(y,x),z)))).
55 [hyper,1,26,28] P(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x))).
70 [hyper,1,3,30] P(i(i(x,y),i(i(i(z,i(u,z)),x),y))).
205 [hyper,1,2,55] P(i(x,i(i(i(y,z),z),i(i(i(y,z),i(z,y)),y)))).
```

907 [hyper,1,50,205] P(i(i(x,y),i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),x),y))).
1891 [hyper,1,23,907] P(i(i(x,y),i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(y,v)),i(x,v)))).
3817 [hyper,1,3,1891] P(i(i(i(i(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x)),i(z,u)),i(v,u)),w),i(i(v,z),w))).
10582 [hyper,1,70,3817] P(i(i(i(x,i(y,x)),i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),
i(v6,w)),v7)),i(i(v6,v),v7))).

For yet another challenge, you might seek a proof of length strictly less than 13 that derives, from the given 4-basis, the Ulrich single axiom. I shall devote a section, Section 6, to shorter proofs.

At this point, you have in hand a proof that the Ulrich 37-symbol formula is a theorem of *MV*-implicational. To complete the study, a proof is needed establishing the deducibility of a known basis from the Ulrich formula. The 4-basis that has been in focus will serve nicely as the target. Rather than giving an appropriate input file, I am content with citing the crucial differences between it and the input file just given for deriving the Ulrich single axiom.

I assigned the value 12 rather than 10 to max_weight; I assigned the value 9 rather than 8 to new_max_weight; and I placed no limit on max_distinct_vars. Of course, in place of the 4-basis in list(sos), I placed the Ulrich formula. As targets, I chose the four members of the basis and the following clause, the latter placed in list(usable).

-P(i(a,i(b,a))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(i(a,b),b),i(i(b,a),a))) | -P(i(i(i(a,b),i(b,a)),i(b,a))) |
$ANS(MVBASISALL).

The clause just given is the negation of the join of the four basis members. My goal, with that clause, was to obtain (in addition to proofs of the individual members) a proof of the join. (Dana Scott many years ago asked for a proof of a set of formulas in which no deduced step occurred more than once; the proof of the just-cited join that was obtained is such a proof, and that is the way to find a proof of the type sought by Scott.) As for the subformula strategy, the same thirteen subformulas were used, which offers the following thought for contemplation. When proving the Ulrich formula, some obvious naturalness is present in that the thirteen are subformulas of the target. At this point, however, the subformulas being used are from the hypothesis, and not the target. The approach worked, and here is the proof of interest.

A 17-Step Proof That Derives from the Ulrich Single Axiom the 4-Basis in Focus

----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on octopus.mcs.anl.gov,
Fri May 30 18:16:07 2008
The command was "otter". The process ID is 28858.

-----> EMPTY CLAUSE at 14.25 sec ----> 2663 [hyper,2,15,64,231,1229] $ANS(MVBASISALL).

Length of proof is 17. Level of proof is 10.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] -P(i(a,i(b,a))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(i(a,b),b),i(i(b,a),a))) | -P(i(i(i(a,b),i(b,a)),i(b,a))) |
$ANS(MVBASISALL).
3 [] P(i(i(i(x,i(y,x)),i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7))).
14 [hyper,1,3,3] P(i(i(x,y),i(z,i(u,z)))).
15 [hyper,1,14,14] P(i(x,i(y,x))).
17 [hyper,1,3,14] P(i(i(x,y),i(z,i(i(i(i(u,v),v),i(i(i(u,v),i(v,u)),u)),i(y,w)),i(x,w))))).
23 [hyper,1,17,15] P(i(x,i(i(i(i(y,z),z),i(i(i(y,z),i(z,y)),y)),i(i(u,v),w)),i(v,w)))).
28 [hyper,1,23,23] P(i(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x)),i(i(z,u),v)),i(u,v))).
34 [hyper,1,28,23] P(i(i(i(x,y),z),i(y,z))).
38 [hyper,1,28,15] P(i(x,i(i(i(y,z),z),i(i(i(y,z),i(z,y)),y)))).

43 [hyper,1,15,34] P(i(x,i(i(i(y,z),u),i(z,u)))).
56 [hyper,1,38,38] P(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x))).
58 [hyper,1,28,38] P(i(x,i(i(i(y,x),i(x,y)),y))).
64 [hyper,1,3,43] P(i(i(x,y),i(i(y,z),i(x,z)))).
102 [hyper,1,64,58] P(i(i(i(i(i(x,y),i(y,x)),x),z),i(y,z))).
103 [hyper,1,64,56] P(i(i(i(i(i(x,y),i(y,x)),x),z),i(i(i(x,y),y),z))).
115 [hyper,1,64,34] P(i(i(i(x,y),z),i(i(i(u,x),y),z))).
231 [hyper,1,103,34] P(i(i(i(x,y),y),i(i(y,x),x))).
302 [hyper,1,115,102] P(i(i(i(x,i(i(i(y,z),i(z,y)),y)),u),i(z,u))).
1229 [hyper,1,102,302] P(i(i(i(x,y),i(y,x)),i(y,x))).

Perhaps you can shorten the given proof. But, at this time, other bases for the implicational fragment of *MV* merit discussion.

4. Other Bases for Implicational MV

You might wonder, especially if you happen to know of an alternative, how things would progress if the 4-basis was replaced by a 3-basis—if such exists—for the implicational fragment of the Lukasiewicz infinite-valued sentential calculus. Indeed, I focus here on two 3-bases, the first from Ulrich and the second from B. Wozniakowska.

```
% Following is Ulrich's 3-basis.
P(i(x,i(y,x))). % simp
P(i(i(x,y),i(i(y,z),i(x,z)))). % syl or B'
P(i(i(i(u,v),v),i(i(i(u,v),i(v,u)),u))). % Ulrich's axiom to replace Linearity and Inversion

% The following is Wozniakowska's 3-basis.
P(i(u,i(v,u))). % Simp
P(i(i(i(u,v),v),i(i(v,u),u))). % Inversion
P(i(i(i(w,x),i(w,y)),i(i(x,w),i(x,y)))). % Wozniakowska's axiom to replace B' and Linearity
```

Whereas the 4-basis featured throughout has, cumulatively, 38 symbols, the Ulrich single axiom 37, each of the just-given bases has (cumulatively) 31 symbols.

In this section, I discuss two studies. My first study, if I recall history correctly, focused on the Ulrich 3-basis. I intended to obtain a proof of the already-given Rezus-style formula for *MV*-implicational and also a proof of the Ulrich 37-symbol single axiom for this area. My plan was to then switch from focusing on the Ulrich 3-basis to focusing on the Wozniakowska 3-basis, with the same two cited goals. Immediately you might ask about my plan to use a single axiom for *MV*-implicational and seek proofs of the two 3-bases. I have not as yet made either study, nor do I know when or if I will. Therefore, perhaps after reading this section, you will (possibly relying on material presented earlier in this notebook) seek the appropriate proofs. Should you, say, prove the Ulrich 3-basis from the Ulrich single axiom, you might then enjoy trying to find a shorter proof for the theorem.

The approach I took to proving from the Ulrich 3-basis the Rezus-style single axiom was reminiscent of the approach I took when studying the 4-basis given earlier. Of course, I replaced the 4-basis by the Ulrich 3-basis. I also assigned the value 9 to max_weight, and I included for monitoring the first 72 steps, negated, of a 73-step proof that derives the Rezus formula from the 4-basis. No, I did not expect to find a proof that closely imitated the 73-step proof. Rather, I include such so-called intermediate targets to see what is happening—perhaps more accurately, to cope with my impatience. I now give the input file, in part to save you from page turning.

An Input File for Deriving Rezus from the Ulrich 3-Basis

```
set(hyper_res).
assign(max_mem,880000).
```

```
% assign(max_seconds,3).
% set(sos_queue).
set(input_sos_first).
assign(max_weight,9).
% assign(change_limit_after,200).
% assign(new_max_weight,8).
assign(max_proofs,-1).
assign(max_distinct_vars,17).
assign(pick_given_ratio,1).
assign(bsub_hint_wt,1).
assign(report,5400).
set(keep_hint_subsumers).
% set(keep_hint_equivalents).
% set(ancestor_subsume).
set(back_sub).
set(order_history).
% set(process_input).
clear(print_kept).

weight_list(pick_and_purge).
% Following 25, including target, are subformulas of the Rezus for the implicational of MV,
% from Ross.
weight(i(v,u),2).
weight(i(x,i(y,x)),2).
weight(i(w,i(v6,w)),2).
weight(i(i(v8,v9),v9),2).
weight(i(i(v9,v8),v8),2).
weight(i(i(v,z),i(v,u)),2).
weight(i(v14,i(v15,v14)),2).
weight(i(i(w,i(v6,w)),v7),2).
weight(i(i(v11,v12),i(v12,v11)),2).
weight(i(i(v14,i(v15,v14)),v16),2).
weight(i(i(z,u),i(i(v,z),i(v,u))),2).
weight(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),2).
weight(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),2).
weight(i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10),2).
weight(i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13),2).
weight(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),2).
weight(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),2).
weight(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),2).
weight(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),2).
weight(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),
  i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),2).
weight(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),
  i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),2).
weight(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16)),2).
weight(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),2).
weight(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),i(i(v14,
  i(v15,v14)),v16))),v16),2).
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weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),i(i(v14,
  i(v15,v14)),v16))),v16)),2).  % Rezus-style axiom for MV
end_of_list.

list(usable).
-P(i(x,y)) | -P(x) | P(y).
%  -P(i(a,a)) | -P(i(i(a,i(b,c)),i(b,i(a,c)))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(a,i(a,b)),i(a,b))) |
  $ANS(all).
end_of_list.

list(sos).
%  Following Ulrich's 3-basis.
P(i(x,i(y,x))).  % Simp, MV1
P(i(i(x,y),i(i(y,z),i(x,z)))).  % Syl, MV2
P(i(i(i(u,v),v),i(i(i(u,v),i(v,u)),u))).  % Ulrich's axiom to replace Linearity and Inversion
end_of_list.

list(passive).
%  Following negs of first 72 of a 73-step proof of Rezus from MV1235.
-P(i(a1,i(a2,i(a3,a2)))) | $ANS(PRF).
-P(i(i(i(i(a1,a2),i(a3,a2)),a4),i(i(a3,a1),a4))) | $ANS(PRF).
-P(i(a1,i(i(a2,a3),i(i(a3,a4),i(a2,a4))))) | $ANS(PRF).
-P(i(i(i(a1,a2),a3),i(a2,a3))) | $ANS(PRF).
-P(i(a1,i(i(i(a2,a3),a3),i(i(a3,a2),a2)))) | $ANS(PRF).
-P(i(a1,i(i(i(a2,a3),i(a3,a2)),i(a3,a2)))) | $ANS(PRF).
-P(i(i(i(a1,i(a2,a1)),a3),a3)) | $ANS(PRF).
-P(i(i(i(i(a1,a2),i(i(a2,a3),i(a1,a3))),a4),a4)) | $ANS(PRF).
-P(i(i(a1,a1),i(a1,a1))) | $ANS(PRF).
-P(i(i(a1,a2),i(i(i(a1,a3),a4),i(i(a2,a3),a4)))) | $ANS(PRF).
-P(i(a1,a1)) | $ANS(PRF).
-P(i(i(i(a1,a2),a3),i(i(i(a4,a1),a2),a3))) | $ANS(PRF).
-P(i(a1,i(i(a1,a2),a2))) | $ANS(PRF).
-P(i(a1,i(a2,a2))) | $ANS(PRF).
-P(i(i(i(i(i(a1,a2),a2),i(i(a2,a1),a1)),a3),a3)) | $ANS(PRF).
-P(i(i(i(a1,a1),a2),a2)) | $ANS(PRF).
-P(i(i(i(i(i(a1,a2),i(a2,a1)),i(a2,a1)),a3),a3)) | $ANS(PRF).
-P(i(i(a1,a2),i(a1,i(a3,a2)))) | $ANS(PRF).
-P(i(i(a1,a2),i(i(i(a3,i(a4,a3)),a1),a2))) | $ANS(PRF).
-P(i(i(i(i(a1,a2),a2),a3),i(a1,a3))) | $ANS(PRF).
-P(i(i(i(a1,i(a2,a2)),a3),a3)) | $ANS(PRF).
-P(i(i(a1,i(a2,a3)),i(a1,i(i(a3,a4),i(a2,a4))))) | $ANS(PRF).
-P(i(i(a1,a2),i(i(i(a3,a3),a1),a2))) | $ANS(PRF).
-P(i(a1,i(i(i(a2,a3),a4),i(i(a1,a3),a4)))) | $ANS(PRF).
-P(i(i(i(i(i(a1,a2),a3),i(i(a4,a2),a3)),b),i(i(a1,a4),b))) | $ANS(PRF).
-P(i(i(a1,i(i(a2,a3),i(a3,a2))),i(a1,i(a3,a2)))) | $ANS(PRF).
-P(i(i(a1,a2),i(i(i(a3,i(a4,a4)),a1),a2))) | $ANS(PRF).
-P(i(i(i(i(i(a1,a2),a3),i(i(a4,a2),a3)),b),i(a4,b))) | $ANS(PRF).
-P(i(i(a1,a2),i(i(a2,a3),i(i(a3,a4),i(a1,a4))))) | $ANS(PRF).
-P(i(i(a1,a2),i(a1,i(i(a2,a3),a3)))) | $ANS(PRF).
-P(i(i(a1,i(a2,a3)),i(a2,i(a1,a3)))) | $ANS(PRF).
-P(i(i(a1,a2),i(i(a3,a1),i(a3,a2)))) | $ANS(PRF).
-P(i(i(a1,i(a2,a3)),i(a1,i(i(a4,a2),i(a4,a3))))) | $ANS(PRF).
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-P(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),a4),a4)) | $ANS(PRF).
-P(i(i(i(i(a1,a2),i(a1,a3)),a4),i(i(a2,a3),a4))) | $ANS(PRF).
-P(i(i(a1,i(i(a2,i(a3,a3)),a4)),i(a1,a4))) | $ANS(PRF).
-P(i(i(a1,i(i(i(i(a2,a3),i(a3,a2)),i(a3,a2)),a4)),i(a1,a4))) | $ANS(PRF).
-P(i(i(a1,i(i(i(i(a2,a3),a3),i(i(a3,a2),a2)),a4)),i(a1,a4))) | $ANS(PRF).
-P(i(i(a1,i(i(a2,i(a3,a2)),a4)),i(a1,a4))) | $ANS(PRF).
-P(i(i(i(a1,a2),a3),i(a4,i(i(a4,a2),a3)))) | $ANS(PRF).
-P(i(i(a1,a2),i(i(i(a3,a2),a4),i(i(a3,a1),a4)))) | $ANS(PRF).
-P(i(i(i(a1,a1),i(a2,a3)),i(i(a3,a4),i(i(a4,b),i(a2,b))))) | $ANS(PRF).
-P(i(i(a1,i(i(a2,a1),a2)),i(a1,a2))) | $ANS(PRF).
-P(i(i(i(a1,a2),i(a3,a4)),i(a3,i(a2,a4)))) | $ANS(PRF).
-P(i(i(i(a1,i(a2,a3)),a4),i(i(a2,i(a1,a3)),a4))) | $ANS(PRF).
-P(i(i(i(i(i(a1,a2),a3),i(i(a1,a4),a3)),b),i(i(a4,a2),b))) | $ANS(PRF).
-P(i(i(i(i(a1,a2),i(a1,a3)),i(a3,a2)),i(a3,a2))) | $ANS(PRF).
-P(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,b)),a6)),a6)) | $ANS(PRF).
-P(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,b)),a6)),i(i(a6,b6),b6))) | $ANS(PRF).
-P(i(i(i(a1,a1),i(i(a2,a3),i(a4,b))),i(a4,i(a3,b)))) | $ANS(PRF).
-P(i(i(a1,i(a2,a3)),i(a2,i(i(a4,a1),i(a4,a3))))) | $ANS(PRF).
-P(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6)) | $ANS(PRF).
-P(i(i(i(i(a1,a2),i(i(a2,a3),i(a1,a3))),i(i(a4,i(b,a4)),a6)),a6)) | $ANS(PRF).
-P(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),b6),b6)) | $ANS(PRF).
-P(i(i(i(i(i(i(a1,a2),i(i(a2,a3),i(a1,a3))),i(i(a4,i(b,a4)),a6)),a6),b6),b6)) | $ANS(PRF).
-P(i(i(i(i(i(a1,a2),a3),i(i(a1,a4),a3)),b),i(a6,i(i(a4,a2),b)))) | $ANS(PRF).
-P(i(a1,i(i(i(i(a2,a3),i(a2,a1)),i(a1,a3)),a3))) | $ANS(PRF).
-P(i(i(i(a1,i(a2,a2)),a3),i(i(i(i(a4,b),i(a4,a3)),i(a3,b)),b))) | $ANS(PRF).
-P(i(i(i(i(i(a1,a1),a2),i(i(a3,a4),a2)),b),b)) | $ANS(PRF).
-P(i(i(a1,i(i(i(i(a2,a2),a3),i(i(a4,b),a3)),a6)),i(a1,a6))) | $ANS(PRF).
-P(i(i(i(a1,a2),i(i(a3,a3),i(a4,b))),i(a4,i(a2,b)))) | $ANS(PRF).
-P(i(a1,i(i(a1,a2),i(i(i(i(a3,a4),i(a3,a2)),i(a2,a4)),a4)))) | $ANS(PRF).
-P(i(a1,i(i(i(i(a1,a2),i(a3,a3)),i(a1,a2)),a2))) | $ANS(PRF).
-P(i(i(i(i(a1,a2),i(a3,a3)),i(a1,a2)),i(i(a4,a1),i(a4,a2)))) | $ANS(PRF).
-P(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(i(b6,b7),b7),i(i(b7,b6),b6)),b8)),b8)) |
  $ANS(PRF).
-P(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(i(b6,b7),b7),
  i(i(b7,b6),b6)),b8)),b8),b9),b9)) | $ANS(PRF).
-P(i(i(i(i(i(i(a1,a2),i(i(a2,a3),i(a1,a3))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(i(b6,b6),b7),
  i(i(b8,b9),b7)),b10)),b10)) | $ANS(PRF).
-P(i(a1,i(i(i(i(a2,a3),i(i(a4,b),i(a4,b))),i(i(a6,i(b6,a6)),b7)),b7))) | $ANS(PRF).
-P(i(i(a1,i(i(i(a2,a3),i(i(a4,b),i(a4,b))),i(i(a6,i(b6,a6)),b7))),i(a1,b7))) | $ANS(PRF).
-P(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(i(b6,b7),b7),
  i(i(b7,b6),b6)),b8)),b8),i(i(i(i(b9,b10),i(b10,b9)),i(b10,b9)),b11)),b11)) | $ANS(PRF).
-P(i(i(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(i(b6,b7),b7),
  i(i(b7,b6),b6)),b8)),b8),i(i(i(i(b9,b10),i(b10,b9)),i(b10,b9)),b11)),b11),b12),i(i(i(i(b13,b14),
  i(b13,b12)),i(b12,b14)),b14))) | $ANS(PRF).
-P(i(i(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(i(b6,b7),b7),
  i(i(b7,b6),b6)),b8)),b8),i(i(i(i(b9,b10),i(b10,b9)),i(b10,b9)),b11)),b11),i(i(b12,
  i(b13,b12)),b14)),b14)) | $ANS(PRF).
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  (i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZPROOF).
% Following is neg of Ulrich's 37-symbol single axiom for the implicational fragment of MV.
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7)),
  i(i(b6,b),b7))) | $ANS(UL37).
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% Following is neg of the rezus-style single axiom for the implicational fragment of MV.
-(P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),i(i(i(i(b8,b9),b9),
  i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)),2) | $ANS(REZMV).
end_of_list.

list(demodulators).
% (i(i(x,x),y) = junk).
% (i(y,i(x,x)) = junk).
(i(x,junk) = junk).
(i(junk,x) = junk).
(P(junk) = $T).
end_of_list.
```

I had no idea how long I would have to wait before I could view the desired proof; in fact, perhaps no proof would be forthcoming. In particular, with three formulas to rely on at the beginning, rather than four, perhaps OTTER would find the task far more difficult. (As an aside, a deep research topic asks about which properties present in an axiom set lead to shorter proofs or lead to quicker proofs; for example, perhaps the number of axioms provides a key, or perhaps the length of the various members, or perhaps some as-yet unspecified property.) The program did succeed, giving more evidence of the value of the subformula strategy, returning the following.

A Proof Deriving the Rezus Formula from the Ulrich 3-Basis

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on elephant.mcs.anl.gov,
Thu May 29 17:11:45 2008
The command was "otter". The process ID is 29594.

----> UNIT CONFLICT at 22113.68 sec ----> 424880 [binary,424879.1,77.1] $ANS(REZPROOF).

Length of proof is 54. Level of proof is 13.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(x,i(y,x))).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(i(u,v),v),i(i(i(u,v),i(v,u)),u))).
77 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZPROOF).
83 [hyper,1,2,2] P(i(x,i(y,i(z,y)))).
85 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
89 [hyper,1,3,2] P(i(i(i(x,y),z),i(y,z))).
91 [hyper,1,3,83] P(i(i(i(x,i(y,x)),z),i(u,z))).
93 [hyper,1,85,85] P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))).
94 [hyper,1,85,3] P(i(i(x,y),i(i(i(x,z),u),i(i(y,z),u)))).
96 [hyper,1,85,2] P(i(i(x,y),i(z,i(i(y,u),i(x,u))))).
99 [hyper,1,3,89] P(i(i(i(x,y),z),i(i(i(u,x),y),z))).
102 [hyper,1,89,4] P(i(x,i(i(i(y,x),i(x,y)),y))).
103 [hyper,1,89,3] P(i(x,i(i(x,y),i(z,y)))).
104 [hyper,1,89,2] P(i(x,i(y,i(z,x)))).
```

118 [hyper,1,89,94] P(i(x,i(i(i(y,z),u),i(i(x,z),u)))).
126 [hyper,1,93,103] P(i(i(x,i(y,z)),i(y,i(x,i(u,z))))).
130 [hyper,1,3,118] P(i(i(i(i(i(x,y),z),i(i(u,y),z)),v),i(u,v))).
159 [hyper,1,99,3] P(i(i(i(x,y),z),i(i(z,u),i(y,u)))).
166 [hyper,1,130,2] P(i(x,i(y,i(i(i(z,u),v),i(i(x,u),v))))).
169 [hyper,1,93,102] P(i(i(x,i(i(y,z),i(z,y))),i(z,i(x,y)))).
219 [hyper,1,126,102] P(i(i(i(x,y),i(y,x)),i(y,i(z,x)))).
224 [hyper,1,159,219] P(i(i(i(x,i(y,z)),u),i(i(x,z),u))).
348 [hyper,1,224,169] P(i(i(x,i(y,z)),i(y,i(x,z)))).
364 [hyper,1,169,166] P(i(i(i(x,y),z),i(u,i(i(u,y),z)))).
369 [hyper,1,169,104] P(i(x,i(y,y))).
372 [hyper,1,169,96] P(i(i(x,y),i(i(z,x),i(z,y)))).
375 [hyper,1,169,2] P(i(x,i(i(x,y),y))).
377 [hyper,1,369,369] P(i(x,x)).
417 [hyper,1,372,91] P(i(i(x,i(i(y,i(z,y)),u)),i(x,i(v,u)))).
418 [hyper,1,372,89] P(i(i(x,i(i(y,z),u)),i(x,i(z,u)))).
456 [hyper,1,3,375] P(i(i(i(i(x,y),y),z),i(x,z))).
458 [hyper,1,375,377] P(i(i(i(x,x),y),y)).
461 [hyper,1,375,372] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)).
463 [hyper,1,375,369] P(i(i(i(x,i(y,y)),z),z)).
470 [hyper,1,375,3] P(i(i(i(i(x,y),i(i(y,z),i(x,z))),u),u)).
472 [hyper,1,375,2] P(i(i(i(x,i(y,x)),z),z)).
508 [hyper,1,372,458] P(i(i(x,i(i(y,y),z)),i(x,z))).
526 [hyper,1,372,463] P(i(i(x,i(i(y,i(z,z)),u)),i(x,u))).
543 [hyper,1,3,472] P(i(i(x,y),i(i(i(z,i(u,z)),x),y))).
574 [hyper,1,3,348] P(i(i(i(x,i(y,z)),u),i(i(y,i(x,z)),u))).
602 [hyper,1,348,102] P(i(i(i(x,y),i(y,x)),i(y,x))).
610 [hyper,1,375,602] P(i(i(i(i(i(x,y),i(y,x)),i(y,x)),z),z)).
671 [hyper,1,372,610] P(i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),i(x,u))).
3091 [hyper,1,417,470] P(i(i(i(i(x,y),i(i(y,z),i(x,z))),i(i(u,i(v,u)),w)),i(v6,w))).
3093 [hyper,1,417,461] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),i(v6,w))).
3108 [hyper,1,418,4] P(i(i(i(x,y),y),i(i(y,x),x))).
3121 [hyper,1,375,3108] P(i(i(i(i(i(x,y),y),i(i(y,x),x)),z),z)).
3218 [hyper,1,574,3121] P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))).
4908 [hyper,1,508,3091] P(i(i(i(i(x,y),i(i(y,z),i(x,z))),i(i(u,i(v,u)),w)),w)).
4952 [hyper,1,364,4908] P(i(x,i(i(x,i(i(y,i(z,y)),u)),u))).
5334 [hyper,1,526,3093] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
5375 [hyper,1,375,5334] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)).
8051 [hyper,1,456,671] P(i(x,i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),u))).
224702 [hyper,1,3218,5375] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8)).
422714 [hyper,1,8051,224702] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)).
422981 [hyper,1,4952,422714] P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w), i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11), i(i(v12,i(v13,v12)),v14)),v14)).
424879 [hyper,1,543,422981] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7), i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13), i(i(v14,i(v15,v14)),v16))),v16)).

You could naturally ask about the relation of this 54-step proof to the 73-step proof given earlier. McCune again played a nice role; indeed, he gave me years ago a program that returns a set-theoretic difference between two sets of one's choosing. As it turned out—in part, in reference to those 72 steps for monitoring—the proof OTTER completed, based solely on the subformula strategy, has length 54 and contains 21 steps not among the 73.

Next in order for discussion is the approach I took to prove the Ulrich single axiom from his 3-basis. As predictable, I borrowed heavily from the preceding input file. The important modifications are the following.

```
assign(max_distinct_vars,8).
%  Following 13 are subformulas  for MV, from Ulrich.
weight(i(u,z),1).
weight(i(i(z,u),u),1).
weight(i(x,i(y,x)),1).
weight(i(i(v6,v),v7),1).
weight(i(i(z,u),i(u,z)),1).
weight(i(i(i(z,u),i(u,z)),z),1).
weight(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),1).
weight(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),1).
weight(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),1).
weight(i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7),1).
weight(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),1).
weight(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7)),1).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7))),1).
% Ulrich 37-symbol for MV-imp
```

In that the target in this case is far simpler (less complex) than is the Rezus formula, I conjectured (not profoundly) that OTTER would return to me the desired proof in far less CPU time. I also expected a proof of length rather less than 54, the length of the proof deriving Rezus from the 3-basis. As you see from the next bit of text, especially regarding proof length, a a nice result was obtained.

A Most Satisfying Proof Deriving the Ulrich 37-Symbol Single Axiom from His 3-Basis

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on octopus.mcs.anl.gov,
Thu Jun  5 18:20:49 2008
The command was "otter".  The process ID is 2500.

----> UNIT CONFLICT at 791.94 sec ----> 105047 [binary,105046.1,15.1] $ANS(UL37PROOF).

Length of proof is 12.  Level of proof is 5.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(x,i(y,x))).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(i(u,v),v),i(i(i(u,v),i(v,u)),u))).
15 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7)),
  i(i(b6,b),b7))) | $ANS(UL37PROOF).
20 [hyper,1,2,2] P(i(x,i(y,i(z,y)))).
21 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
23 [hyper,1,3,2] P(i(i(i(x,y),z),i(y,z))).
24 [hyper,1,3,4] P(i(i(i(i(i(x,y),i(y,x)),x),z),i(i(i(x,y),y),z))).
25 [hyper,1,2,4] P(i(x,i(i(i(y,z),z),i(i(i(y,z),i(z,y)),y)))).
28 [hyper,1,21,21] P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))).
```

43 [hyper,1,24,23] P(i(i(i(x,y),y),i(i(y,x),x))).
94 [hyper,1,43,25] P(i(i(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x)),z),z)).
96 [hyper,1,43,20] P(i(i(i(x,i(y,x)),z),z)).
1530 [hyper,1,28,94] P(i(i(x,y),i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(y,v)),i(x,v)))).
1606 [hyper,1,28,96] P(i(i(x,y),i(i(i(z,i(u,z)),i(y,v)),i(x,v)))).
105046 [hyper,1,1606,1530] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)), i(i(v6,v),v7))).

A glance at this proof, because of its brevity, naturally suggests a possible connection between his 3-basis and his single axiom. When I mentioned this point to Ulrich, he more than confirmed a sharp connection. Indeed, the last three lines of the 12-step proof (in effect) show how Ulrich found his single axiom. The first two of the three, steps 10 and 11, he produced by hand, by inserting various formulas at key points. The last of the three, step 12, is obtained by applying condensed detachment to the tenth and eleventh steps of the 12-step proof. Put succinctly, steps 10, 11, and 12 capture in miniature Ulrich's research that led to his finding that splendid 37-symbol single axiom for *MV*-imp.

As noted near the beginning of this section, my plan called for me now to switch to the study of the Wozniakowska 3-basis, which I repeat almost immediately. I modified (if memory serves) the input file I gave earlier, that which derives Rezus from the Ulrich 3-basis, with the following seven lines.

```
assign(max_weight,10).
assign(change_limit_after,200).
assign(new_max_weight,8).
% Following Wozniakowska's 3-basis
P(i(u,i(v,u))).   % Simp K
P(i(i(i(u,v),v),i(i(v,u),u))). % Inversion
P(i(i(i(w,x),i(w,y)),i(i(x,w),i(x,y)))). % Wozniakowska's axiom to replace B' and Linearity
```

The last four of these seven lines, in place of its Ulrich counterpart, are placed in list(sos), which has the program study the Wozniakowska 3-basis. The first three lines instruct OTTER about the weight (complexity) of retained clauses and how to proceed. I had tried two experiments that precede that under discussion, each relying on other treatments of weight templates; both had failed. But, because experimentation is made (at least for me) so exciting, I finally hit on the right set of options and values to assign to diverse parameters. The sought-after proof is the following.

A Proof of Rezus from the Wozniakowska 3-Basis Relying on the Subformula Strategy Alone

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on octopus.mcs.anl.gov,
Sat May 31 09:18:41 2008
The command was "otter". The process ID is 28345.

----> UNIT CONFLICT at 14348.18 sec ----> 111354 [binary,111353.1,77.1] $ANS(REZPROOF).

Length of proof is 59. Level of proof is 14.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(u,i(v,u))).
3 [] P(i(i(i(u,v),v),i(i(v,u),u))).
4 [] P(i(i(i(w,x),i(w,y)),i(i(x,w),i(x,y)))).
77 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZPROOF).
```

83 [hyper,1,2,2] P(i(x,i(y,i(z,y)))).
85 [hyper,1,2,3] P(i(x,i(i(i(y,z),z),i(i(z,y),y)))).
88 [hyper,1,2,4] P(i(x,i(i(i(y,z),i(y,u)),i(i(z,y),i(z,u))))).
90 [hyper,1,4,83] P(i(i(x,y),i(x,i(z,y)))).
92 [hyper,1,3,83] P(i(i(i(x,i(y,x)),z),z)).
95 [hyper,1,4,85] P(i(i(x,i(i(y,z),z)),i(x,i(i(z,y),y)))).
96 [hyper,1,3,85] P(i(i(i(i(i(x,y),y),i(i(y,x),x)),z),z)).
99 [hyper,1,2,92] P(i(x,i(i(i(y,i(z,y)),u),u))).
103 [hyper,1,2,96] P(i(x,i(i(i(i(i(y,z),z),i(i(z,y),y)),u),u))).
105 [hyper,1,4,88] P(i(i(x,i(i(y,z),i(y,u))),i(x,i(i(z,y),i(z,u))))).
106 [hyper,1,3,88] P(i(i(i(i(i(x,y),i(x,z)),i(i(y,x),i(y,z))),u),u)).
113 [hyper,1,96,90] P(i(i(i(x,y),y),i(z,i(i(y,x),x)))).
132 [hyper,1,92,95] P(i(x,i(i(x,y),y))).
153 [hyper,1,4,99] P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))).
159 [hyper,1,4,103] P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))).
162 [hyper,1,132,132] P(i(i(i(x,i(i(x,y),y)),z),z)).
164 [hyper,1,90,132] P(i(x,i(y,i(i(x,z),z)))).
171 [hyper,1,92,105] P(i(i(x,y),i(i(z,x),i(z,y)))).
178 [hyper,1,171,171] P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))).
180 [hyper,1,132,171] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)).
190 [hyper,1,171,90] P(i(i(x,i(y,z)),i(x,i(y,i(u,z))))).
201 [hyper,1,105,113] P(i(i(i(x,y),y),i(i(z,i(y,x)),i(z,x)))).
378 [hyper,1,171,153] P(i(i(x,i(y,i(i(z,i(u,z)),v))),i(x,i(y,v)))).
382 [hyper,1,106,153] P(i(i(i(i(x,y),y),i(i(x,y),z)),i(y,z))).
390 [hyper,1,153,171] P(i(i(i(x,y),z),i(y,z))).
398 [hyper,1,171,382] P(i(i(x,i(i(i(y,z),z),i(i(y,z),u))),i(x,i(z,u)))).
411 [hyper,1,95,390] P(i(i(i(x,i(y,z)),z),i(i(z,y),y))).
436 [hyper,1,159,171] P(i(i(i(i(x,y),y),z),i(i(i(y,x),x),z))).
483 [hyper,1,105,164] P(i(x,i(i(y,i(x,z)),i(y,z)))).
504 [hyper,1,171,180] P(i(i(x,i(i(i(y,z),i(i(u,y),i(u,z))),v)),i(x,v))).
506 [hyper,1,153,180] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
515 [hyper,1,171,506] P(i(i(x,i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6))),i(x,v6))).
519 [hyper,1,132,506] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)).
693 [hyper,1,159,519] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8)).
708 [hyper,1,162,190] P(i(x,i(i(x,y),i(z,y)))).
723 [hyper,1,190,411] P(i(i(i(x,i(y,z)),z),i(i(z,y),i(u,y)))).
751 [hyper,1,132,693] P(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8),v9),v9)).
899 [hyper,1,171,723] P(i(i(x,i(i(y,i(z,u)),u)),i(x,i(i(u,z),i(v,z))))).
1436 [hyper,1,180,378] P(i(i(x,y),i(i(i(z,i(u,z)),x),y))).
1467 [hyper,1,190,483] P(i(x,i(i(y,i(x,z)),i(u,i(y,z))))).
1494 [hyper,1,398,436] P(i(i(i(i(x,y),y),i(i(y,x),z)),i(x,z))).
1503 [hyper,1,171,1494] P(i(i(x,i(i(i(y,z),z),i(i(z,y),u))),i(x,i(y,u)))).
1527 [hyper,1,178,515] P(i(i(x,i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6))),i(i(v7,x),i(v7,v6)))).
1562 [hyper,1,504,483] P(i(i(x,y),i(i(y,z),i(x,z)))).
1564 [hyper,1,504,201] P(i(i(i(i(x,y),i(x,z)),i(x,z)),i(i(z,y),i(x,y)))).
1566 [hyper,1,1562,1562] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
1573 [hyper,1,708,1562] P(i(i(i(i(x,y),i(i(y,z),i(x,z))),u),i(v,u))).
1593 [hyper,1,171,1562] P(i(i(x,i(y,z)),i(x,i(i(z,u),i(y,u))))).
5833 [hyper,1,1503,1562] P(i(i(i(x,y),i(y,x)),i(y,x))).
5848 [hyper,1,1467,5833] P(i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),i(v,i(x,u)))).
6314 [hyper,1,1566,899] P(i(i(i(x,i(y,z)),u),i(i(u,z),i(i(z,y),i(v,y))))).

6329 [hyper,1,1527,1573] P(i(i(x,i(i(i(y,z),i(i(z,u),i(y,u))),i(i(v,i(w,v)),v6))),i(x,v6))).
6576 [hyper,1,1593,1564] P(i(i(i(i(x,y),i(x,z)),i(x,z)),i(i(i(x,y),u),i(i(z,y),u)))).
9558 [hyper,1,436,6576] P(i(i(i(i(x,y),i(x,z)),i(x,z)),i(i(i(x,z),u),i(i(y,z),u)))).
14260 [hyper,1,6314,9558] P(i(i(i(i(i(x,y),z),i(i(u,y),z)),y),i(i(y,x),i(v,x)))).
23965 [hyper,1,5848,751] P(i(x,i(i(i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6),i(i(i(v7,v8),v8), i(i(v8,v7),v7)),v9)),v9),i(i(i(v10,v11),i(v11,v10)),i(v11,v10)),v12)),v12))).
24236 [hyper,1,14260,23965] P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w), i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),v12), i(v13,v12))).
111295 [hyper,1,6329,24236] P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w), i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11), i(i(v12,i(v13,v12)),v14)),v14)).
111353 [hyper,1,1436,111295] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))), i(i(w,i(v6,w)),v7)),v7),i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(v11,v12),i(v12,v11)), i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)).

Although this proof is five steps longer than the 54-step proof deducing Rezus from the Ulrich 3-basis, it was completed in much less CPU time. Again, as indicated earlier in the discussion of a deep research topic, I wonder if the explanation rests with properties of the Wozniakowska 3-basis, rather than fully with the options and parameters that were used. This burning question might be more puzzling when you learn what happened with the following, namely, when the goal of proving Rezus with the Wozniakowska 3-basis was replaced with proving the Ulrich 37-symbol single axiom.

The following changes were made to the input file, that led to a derivation of Rezus from the Wozniakowska 3-basis, in order to seek a derivation of the Ulrich single axiom.

```
% assign(max_seconds,2).
assign(max_weight,9).
% assign(change_limit_after,100).
% assign(new_max_weight,8).
assign(max_distinct_vars,8).
```

Other than the instructions that are commented out, the crucial change here is the assignment of the value 8 to max_distinct_vars. That value was chosen because the Ulrich axiom relies on precisely eight distinct variables. In addition, as dictated by the subformula strategy, the following subformulas were used in place of those that are themselves relevant to the Rezus axiom.

```
% Following 13 are subformulas of the Ulrich 37 for MV, from Overbeek.
weight(i(u,z),1).
weight(i(i(z,u),u),1).
weight(i(x,i(y,x)),1).
weight(i(i(v6,v),v7),1).
weight(i(i(z,u),i(u,z)),1).
weight(i(i(i(z,u),i(u,z)),z),1).
weight(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),1).
weight(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),1).
weight(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),1).
weight(i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7),1).
weight(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),1).
weight(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7)),1).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7))),1).
% Ulrich 37-symbol for MV-imp
```

When the modified input file was presented to OTTER, eventually, the program returned to me the desired proof.

A Proof Deriving the Ulrich 37-Symbol Single Axiom from the Wozniakowska 3-Basis

----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on octopus.mcs.anl.gov,
Thu Jun 5 18:44:21 2008
The command was "otter". The process ID is 2571.

----> UNIT CONFLICT at 5924.39 sec ----> 266645 [binary,266644.1,15.1] $ANS(UL37PROOF).

Length of proof is 73. Level of proof is 26.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(u,i(v,u))).
3 [] P(i(i(i(u,v),v),i(i(v,u),u))).
4 [] P(i(i(i(w,x),i(w,y)),i(i(x,w),i(x,y)))).
15 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7)), i(i(b6,b),b7))) | $ANS(UL37PROOF).
20 [hyper,1,2,2] P(i(x,i(y,i(z,y)))).
21 [hyper,1,2,3] P(i(x,i(i(i(y,z),z),i(i(z,y),y)))).
23 [hyper,1,2,4] P(i(x,i(i(i(y,z),i(y,u)),i(i(z,y),i(z,u))))).
24 [hyper,1,4,3] P(i(i(x,i(x,x)),i(x,x))).
25 [hyper,1,4,20] P(i(i(x,y),i(x,i(z,y)))).
26 [hyper,1,3,20] P(i(i(i(x,i(y,x)),z),z)).
28 [hyper,1,4,21] P(i(i(x,i(i(y,z),z)),i(x,i(i(z,y),y)))).
29 [hyper,1,3,21] P(i(i(i(i(i(x,y),y),i(i(y,x),x)),z),z)).
35 [hyper,1,4,23] P(i(i(x,i(i(y,z),i(y,u))),i(x,i(i(z,y),i(z,u))))).
36 [hyper,1,3,23] P(i(i(i(i(i(x,y),i(x,z)),i(i(y,x),i(y,z))),u),u)).
40 [hyper,1,24,2] P(i(x,x)).
45 [hyper,1,25,4] P(i(i(i(x,y),i(x,z)),i(u,i(i(y,x),i(y,z))))).
49 [hyper,1,2,40] P(i(x,i(y,y))).
53 [hyper,1,3,49] P(i(i(i(x,x),y),y)).
62 [hyper,1,26,28] P(i(x,i(i(x,y),y))).
71 [hyper,1,25,29] P(i(i(i(i(i(x,y),y),i(i(y,x),x)),z),i(u,z))).
74 [hyper,1,25,53] P(i(i(i(x,x),y),i(z,y))).
75 [hyper,1,62,62] P(i(i(i(x,i(i(x,y),y)),z),z)).
85 [hyper,1,26,35] P(i(i(x,y),i(i(z,x),i(z,y)))).
88 [hyper,1,35,28] P(i(i(i(i(x,y),z),i(i(y,x),x)),i(i(z,i(x,y)),i(z,y)))).
93 [hyper,1,35,74] P(i(i(i(x,x),i(y,z)),i(i(u,y),i(u,z)))).
94 [hyper,1,28,74] P(i(i(i(x,x),y),i(i(y,z),z))).
100 [hyper,1,85,85] P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))).
101 [hyper,1,62,85] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)).
108 [hyper,1,85,53] P(i(i(x,i(i(y,y),z)),i(x,z))).
110 [hyper,1,85,26] P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))).
111 [hyper,1,85,25] P(i(i(x,i(y,z)),i(x,i(y,i(u,z))))).
135 [hyper,1,85,93] P(i(i(x,i(i(y,y),i(z,u))),i(x,i(i(v,z),i(v,u))))).
143 [hyper,1,62,94] P(i(i(i(i(i(x,x),y),i(i(y,z),z)),u),u)).
149 [hyper,1,35,45] P(i(i(i(x,y),i(x,z)),i(i(u,i(y,x)),i(u,i(y,z))))).
150 [hyper,1,28,45] P(i(i(i(x,y),i(x,z)),i(i(i(i(y,x),i(y,z)),u),u))).
193 [hyper,1,35,71] P(i(i(i(i(i(x,y),y),i(i(y,x),x)),i(z,u)),i(i(v,z),i(v,u)))).
214 [hyper,1,100,93] P(i(i(i(x,x),i(y,z)),i(i(u,i(v,y)),i(u,i(v,z))))).
226 [hyper,1,100,36] P(i(i(i(i(i(x,y),i(x,z)),i(i(y,x),i(y,z))),i(u,v)),i(i(w,u),i(w,v)))).
287 [hyper,1,88,88] P(i(i(i(i(x,y),y),i(i(y,x),y)),i(i(i(x,y),y),y))).

309 [hyper,1,108,29] P(i(i(i(i(i(x,y),y),i(i(y,x),x)),i(i(z,z),u)),u)).
339 [hyper,1,75,111] P(i(x,i(i(x,y),i(z,y)))).
389 [hyper,1,62,339] P(i(i(i(x,i(i(x,y),i(z,y))),u),u)).
435 [hyper,1,101,110] P(i(i(i(x,y),z),i(y,z))).
459 [hyper,1,85,435] P(i(i(x,i(i(y,z),u)),i(x,i(z,u)))).
566 [hyper,1,110,149] P(i(i(i(x,y),i(x,z)),i(x,i(y,z)))).
575 [hyper,1,85,566] P(i(i(x,i(i(y,z),i(y,u))),i(x,i(y,i(z,u))))).
612 [hyper,1,135,75] P(i(i(i(x,i(i(x,y),y)),i(i(z,z),i(u,v))),i(i(w,u),i(w,v)))).
707 [hyper,1,214,150] P(i(i(x,i(y,i(i(i(z,u),i(z,z)),v))),i(x,i(y,v)))).
968 [hyper,1,193,110] P(i(i(x,i(i(i(y,z),z),z)),i(x,i(y,z)))).
1159 [hyper,1,459,143] P(i(i(i(i(i(x,x),y),i(i(y,z),z)),i(i(u,v),w)),i(v,w))).
1182 [hyper,1,566,226] P(i(i(i(i(x,y),i(x,z)),i(i(y,x),i(y,z))),i(i(i(i(y,x),i(y,z)),u),i(i(i(x,y),i(x,z)),u)))).
6405 [hyper,1,968,287] P(i(i(i(i(x,y),y),i(i(y,x),y)),i(x,y))).
8287 [hyper,1,36,1182] P(i(i(i(i(x,y),i(x,z)),u),i(i(i(y,x),i(y,z)),u))).
8313 [hyper,1,459,8287] P(i(i(i(i(x,y),i(x,z)),u),i(i(y,z),u))).
8567 [hyper,1,707,8313] P(i(i(i(i(x,y),i(x,z)),i(i(i(u,v),i(u,u)),w)),i(i(y,z),w))).
8631 [hyper,1,8313,1159] P(i(i(i(i(x,y),y),z),i(x,z))).
8669 [hyper,1,8313,389] P(i(i(i(i(x,y),i(z,y)),u),i(x,u))).
8688 [hyper,1,8313,110] P(i(i(x,y),i(i(i(z,i(u,z)),x),y))).
8940 [hyper,1,8669,575] P(i(x,i(i(x,i(y,z)),i(y,i(u,z))))).
14487 [hyper,1,8669,8567] P(i(i(x,y),i(i(y,z),i(x,z)))).
14513 [hyper,1,14487,14487] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
15015 [hyper,1,14513,6405] P(i(i(i(x,y),i(y,x)),i(y,x))).
15948 [hyper,1,14487,15015] P(i(i(i(x,y),z),i(i(i(y,x),i(x,y)),z))).
19405 [hyper,1,14513,8631] P(i(i(x,i(y,z)),i(y,i(x,z)))).
44663 [hyper,1,8940,15948] P(i(i(i(i(i(x,y),z),i(i(i(y,x),i(x,y)),z)),i(u,v)),i(u,i(w,v)))).
44899 [hyper,1,19405,44663] P(i(x,i(i(i(i(i(y,z),u),i(i(i(z,y),i(y,z)),u)),i(x,v)),i(w,v)))).
45237 [hyper,1,14487,44899] P(i(i(i(i(i(i(i(x,y),z),i(i(i(y,x),i(x,y)),z)),i(u,v)),i(w,v)),v6),i(u,v6))).
45421 [hyper,1,45237,612] P(i(i(i(i(i(x,y),z),i(i(i(y,x),i(x,y)),z)),i(u,v)),i(i(w,u),i(w,v)))).
45604 [hyper,1,19405,45421] P(i(i(x,y),i(i(i(i(i(z,u),v),i(i(i(u,z),i(z,u)),v)),i(y,w)),i(x,w)))).
46237 [hyper,1,309,45604] P(i(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x))).
46373 [hyper,1,8940,46237] P(i(i(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x)),i(z,u)),i(z,i(v,u)))).
47332 [hyper,1,19405,46373] P(i(x,i(i(i(i(i(y,z),z),i(i(i(y,z),i(z,y)),y)),i(x,u)),i(v,u)))).
48437 [hyper,1,14487,47332] P(i(i(i(i(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x)),i(z,u)),i(v,u)),w),i(z,w))).
49117 [hyper,1,48437,612] P(i(i(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x)),i(z,u)),i(i(v,z),i(v,u)))).
49533 [hyper,1,19405,49117] P(i(i(x,y),i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(y,v)),i(x,v)))).
49735 [hyper,1,14487,49533] P(i(i(i(i(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x)),i(z,u)),i(v,u)),w),i(i(v,z),w))).
266644 [hyper,1,8688,49735] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),
i(v6,w)),v7)),i(i(v6,v),v7))).

I, as perhaps you do, wonder about the lengthier proof when compared with the proof that derives the Rezus formula. After all, the hypotheses are shared; of course, the target is different, but far less complex. And yet the proof is longer, longer for a less complex target. You might, if curious, investigate what would occur if in these various experiments other sets of subformulas were included, for example, subformulas of hypothesis or target as the case requires. Also, possibly of interest are the experiments I have not yet conducted (in this context) that derive, respectively, the Ulrich 3-basis and the Wozniakowska 3-basis from the Rezus single axiom and then from the Ulrich single axiom for *MV*-imp.

At this point, I have completed (for this notebook) my focus on the implicational fragment of the Lukasiewicz infinite-valued sentential calculus. In a later section, I return to the study of *MV*-implicational in the context of proof shortening. In keeping with one of the many goals I have in these notebooks, you might offer a topic for investigation and compare your suggestion with the choice I made, given in the following section.

5. Some Studies of MV as a Whole

Earlier you were presented with five axioms for the Lukasiewicz infinite-valued sentential calculus, those he originally offered. Thanks to Meredith, the fifth (which I denote as *MV*5) was shown to be dependent on *MV*1 through *MV*4. The fourth axiom, the following, introduces the function *n* for negation.

```
P(i(i(n(x),n(y)),i(y,x))).  % MV4
```

My chosen research topic concerns the use and effects of studying the full *MV* in the context of the subformula strategy. Perhaps the presence of two functions, *i* (for implication) and *n* (for negation), would in some unspecified manner diminish the apparent power of the subformula strategy. Perhaps their presence in a target would result in the availability of too many subformulas. These questions did occur to me.

In imitation of what has gone before, I inquired about one or more single axioms for all of *MV*. I learned (from Ulrich) that, at present, a so-called nice single axiom is not known. However, he did send me the following Rezus-style single axiom for the full infinite-valued calculus.

```
P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)).
```

Those who enjoy details and comparisons will see that the function *n* occurs but twice in this single axiom. Also, this axiom closely resembles the Rezus-style single axiom for *MV*-implicational. The not-surprising study asks for a proof that derives the Rezus axiom from the (now-in-focus) 4-basis consisting of *MV*1 through *MV*4. This study also asks for a proof that derives the now-in-focus 4-basis from the Rezus-style axiom for the full *MV* logic. Of course, both bits of research will rely (at first) exclusively on the use of the appropriate subformulas. Rather than proceeding as I have recently by discussing modifications to an earlier input file, here I give the following input file. The file, as you see, contains appropriate subformulas. It also relies on the use of ancestor subsumption, which I will shortly discuss.

An Input File for Deriving in MV-Full a Rezus Single Axiom from a 4-Basis

```
% Trying for a proof of MV5 from MV1-4 avoiding 2.22 3.5 3.51.
set(hyper_res).
assign(max_weight,8).
assign(change_limit_after,200).
assign(new_max_weight,8).
assign(max_proofs,-1).
clear(print_kept).
% set(process_input).
set(ancestor_subsume).
set(back_sub).
clear(print_back_sub).
assign(max_distinct_vars,17).
% assign(pick_given_ratio,4).
% assign(max_seconds,120).
assign(max_mem,480000).
assign(report,3600).
set(order_history).
set(input_sos_first).
set(sos_queue).
assign(heat,0).
% assign(dynamic_heat_weight,0).

weight_list(pick_and_purge).
% Following 24 are subformulas, from Overbeek, of a Rezus-style formula from Ulrich,
%a single axiom for all of MV.
weight(i(v,u),2).
```

```
weight(n(v11),2).
weight(i(x,i(y,x)),2).
weight(i(i(v8,v9),v9),2).
weight(i(i(v,z),i(v,u)),2).
weight(i(n(v11),n(v12)),2).
weight(i(v14,i(v15,v14)),2).
weight(i(i(w,i(v6,w)),v7),2).
weight(i(i(v14,i(v15,v14)),v16),2).
weight(i(i(z,u),i(i(v,z),i(v,u))),2).
weight(i(i(n(v11),n(v12)),i(v12,v11)),2).
weight(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),2).
weight(i(i(i(n(v11),n(v12)),i(v12,v11)),v13),2).
weight(i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10),2).
weight(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),2).
weight(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),2).
weight(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),2).
weight(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),2).
weight(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),2).
weight(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),2).
weight(i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16)),2).
weight(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
  i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),2).
weight(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16),2).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
 yy(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)),2).
end_of_list.

list(usable).
% condensed detachment
-P(i(x,y)) | -P(x) | P(y).
end_of_list.

list(sos).
%  Just the implicational axioms: A1 - A3
P(i(x,i(y,x))).
P(i(i(x,y),i(i(y,z),i(x,z)))).
P(i(i(i(x,y),y),i(i(y,x),x))).
%  Following is MV4.
P(i(i(n(x),n(y)),i(y,x))).
end_of_list.

list(passive).
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),i(i(i(i(b8,b9),b9),
  i(i(b9,b8),b8)),b10)),b10),i(i(i(n(b11),n(b12)),i(b12,b11)),b13)),b13),i(i(b14,i(b15,b14)),b16))),b16)) |
  $ANS(REZMV).
end_of_list.

list(demodulators).
```

```
% (n(n(n(x))) = junk).
(n(n(x)) = junk).
% (i(i(x,x),y) = junk).
% (i(y,i(x,x)) = junk).
% (i(n(i(x,x)),y) = junk).
% (i(y,n(i(x,x))) = junk).
(i(x,junk) = junk).
(i(junk,x) = junk).
(n(junk) = junk).
(P(junk) = $T).
end_of_list.

list(hot).
-P(i(x,y)) | -P(x) | P(y).
% Following is MV1 -- MV4.
P(i(x,i(y,x))).
P(i(i(x,y),i(i(y,z),i(x,z)))).
P(i(i(i(x,y),y),i(i(y,x),x))).
P(i(i(n(x),n(y)),i(y,x))).
end_of_list.
```

I know not why I included the use of ancestor subsumption, other than, perhaps, an impatience to see a proof and then shorter proofs. But, having just glanced at the results of using the input file just given, and noting that the use of ancestor subsumption can cost very much in CPU time, I am at this moment of this writing conducting another experiment. In this experiment, I am avoiding the use of ancestor subsumption, and I am relying on a complexity-preference search rather than a level-saturation search, using the value of 4 assigned to the pick_given_ratio. When that experiment is complete, running on the same computer as that which I now cite in the following proof, I shall present some of the results and, probably and make some comments.

Three Proofs Showing that the Rezus Axiom for MV Is Derivable from the 4-Basis for Full MV

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on elephant.mcs.anl.gov,
Sun Jun 15 12:22:56 2008
The command was "otter". The process ID is 29239.

----> UNIT CONFLICT at 85912.80 sec ----> 287235 [binary,287234.1,6.1] $ANS(REZMV).

Length of proof is 45. Level of proof is 13.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(x,i(y,x))).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(i(x,y),y),i(i(y,x),x))).
5 [] P(i(i(n(x),n(y)),i(y,x))).
6 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(n(b11),n(b12)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZMV).
17 [hyper,1,2,2] P(i(x,i(y,i(z,y)))).
18 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
```

19 [hyper,1,2,3] P(i(x,i(i(y,z),i(i(z,u),i(y,u))))).
20 [hyper,1,3,2] P(i(i(i(x,y),z),i(y,z))).
24 [hyper,1,4,17] P(i(i(i(x,i(y,x)),z),z)).
27 [hyper,1,18,3] P(i(i(x,y),i(i(i(x,z),u),i(i(y,z),u)))).
29 [hyper,1,4,19] P(i(i(i(i(x,y),i(i(y,z),i(x,z))),u),u)).
35 [hyper,1,20,4] P(i(x,i(i(x,y),y))).
36 [hyper,1,20,3] P(i(x,i(i(x,y),i(z,y)))).
47 [hyper,1,20,24] P(i(x,x)).
48 [hyper,1,18,24] P(i(i(x,y),i(x,i(z,y)))).
49 [hyper,1,3,24] P(i(i(x,y),i(i(i(z,i(u,z)),x),y))).
58 [hyper,1,3,27] P(i(i(i(i(i(x,y),z),i(i(u,y),z)),v),i(i(x,u),v))).
68 [hyper,1,18,29] P(i(i(x,i(y,z)),i(x,i(i(z,u),i(y,u))))).
88 [hyper,1,3,35] P(i(i(i(i(x,y),y),z),i(x,z))).
91 [hyper,1,3,36] P(i(i(i(i(x,y),i(z,y)),u),i(x,u))).
126 [hyper,1,35,47] P(i(i(i(x,x),y),y)).
132 [hyper,1,3,48] P(i(i(i(x,i(y,z)),u),i(i(x,z),u))).
253 [hyper,1,18,88] P(i(i(x,i(y,z)),i(y,i(x,z)))).
260 [hyper,1,88,18] P(i(i(x,y),i(i(z,x),i(z,y)))).
366 [hyper,1,3,126] P(i(i(x,y),i(i(i(z,z),x),y))).
539 [hyper,1,253,253] P(i(x,i(i(y,i(x,z)),i(y,z)))).
549 [hyper,1,3,253] P(i(i(i(x,i(y,z)),u),i(i(y,i(x,z)),u))).
607 [hyper,1,91,260] P(i(x,i(i(y,i(x,z)),i(y,i(u,z))))).
613 [hyper,1,35,260] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)).
830 [hyper,1,366,49] P(i(i(i(x,x),i(y,z)),i(i(i(u,i(v,u)),y),z))).
1055 [hyper,1,539,4] P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))).
1056 [hyper,1,539,2] P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))).
1091 [hyper,1,549,132] P(i(i(x,i(i(y,i(z,u)),v)),i(i(y,u),i(x,v)))).
1095 [hyper,1,549,58] P(i(i(x,i(i(i(i(y,z),u),i(i(v,z),u)),w)),i(i(y,v),i(x,w)))).
1252 [hyper,1,68,607] P(i(x,i(i(i(y,i(z,u)),v),i(i(y,i(x,u)),v)))).
2060 [hyper,1,1056,613] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
2111 [hyper,1,1091,29] P(i(i(x,y),i(i(i(i(z,u),i(i(u,v),i(z,v))),i(i(x,i(w,y)),v6)),v6))).
2119 [hyper,1,68,1095] P(i(i(x,i(i(i(i(y,z),u),i(i(v,z),u)),w)),i(i(i(x,w),v6),i(i(y,v),v6)))).
2456 [hyper,1,253,1252] P(i(i(i(x,i(y,z)),u),i(v,i(i(x,i(v,z)),u)))).
4360 [hyper,1,35,2060] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)).
4563 [hyper,1,3,2111] P(i(i(i(i(i(i(x,y),i(i(y,z),i(x,z))),i(i(u,i(v,w)),v6)),v6),v7),i(i(u,w),v7))).
5051 [hyper,1,2456,2119] P(i(x,i(i(y,i(x,z)),i(i(i(y,z),u),i(i(v,w),u))))).
8645 [hyper,1,1055,4360] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8)).
9300 [hyper,1,18,4563] P(i(i(x,i(i(i(y,z),i(i(z,u),i(y,u))),i(i(v,i(w,v6)),v7))),i(i(v,v6),i(x,v7)))).
10242 [hyper,1,5051,5] P(i(i(x,i(i(i(n(y),n(z)),i(z,y)),u)),i(i(i(x,u),v),i(i(w,v6),v)))).
17518 [hyper,1,35,8645] P(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8),v9),v9)).
45108 [hyper,1,10242,17518] P(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w), i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),i(i(i(n(v9),n(v10)),i(v10,v9)),v11)),v11),v12),i(i(v13,v14),v12))).
121126 [hyper,1,9300,45108] P(i(i(x,y),i(i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7), i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13), i(i(x,i(v14,y)),v15)),v15))).
287234 [hyper,1,830,121126] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))), i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)), i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)).

----> UNIT CONFLICT at 85916.54 sec ----> 287255 [binary,287254.1,6.1] $ANS(REZMV).

Length of proof is 41. Level of proof is 13.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(x,i(y,x))).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(i(x,y),y),i(i(y,x),x))).
5 [] P(i(i(n(x),n(y)),i(y,x))).
6 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(n(b11),n(b12)),i(b12,b11)),b13)),b13),
i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZMV).
17 [hyper,1,2,2] P(i(x,i(y,i(z,y)))).
18 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
19 [hyper,1,2,3] P(i(x,i(i(y,z),i(i(z,u),i(y,u))))).
20 [hyper,1,3,2] P(i(i(i(x,y),z),i(y,z))).
24 [hyper,1,4,17] P(i(i(i(x,i(y,x)),z),z)).
27 [hyper,1,18,3] P(i(i(x,y),i(i(i(x,z),u),i(i(y,z),u)))).
29 [hyper,1,4,19] P(i(i(i(i(x,y),i(i(y,z),i(x,z))),u),u)).
35 [hyper,1,20,4] P(i(x,i(i(x,y),y))).
36 [hyper,1,20,3] P(i(x,i(i(x,y),i(z,y)))).
47 [hyper,1,20,24] P(i(x,x)).
49 [hyper,1,3,24] P(i(i(x,y),i(i(i(z,i(u,z)),x),y))).
58 [hyper,1,3,27] P(i(i(i(i(i(x,y),z),i(i(u,y),z)),v),i(i(x,u),v))).
68 [hyper,1,18,29] P(i(i(x,i(y,z)),i(x,i(i(z,u),i(y,u))))).
88 [hyper,1,3,35] P(i(i(i(i(x,y),y),z),i(x,z))).
91 [hyper,1,3,36] P(i(i(i(i(x,y),i(z,y)),u),i(x,u))).
126 [hyper,1,35,47] P(i(i(i(x,x),y),y)).
253 [hyper,1,18,88] P(i(i(x,i(y,z)),i(y,i(x,z)))).
260 [hyper,1,88,18] P(i(i(x,y),i(i(z,x),i(z,y)))).
366 [hyper,1,3,126] P(i(i(x,y),i(i(i(z,z),x),y))).
539 [hyper,1,253,253] P(i(x,i(i(y,i(x,z)),i(y,z)))).
549 [hyper,1,3,253] P(i(i(i(x,i(y,z)),u),i(i(y,i(x,z)),u))).
607 [hyper,1,91,260] P(i(x,i(i(y,i(x,z)),i(y,i(u,z))))).
613 [hyper,1,35,260] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)).
830 [hyper,1,366,49] P(i(i(i(x,x),i(y,z)),i(i(i(u,i(v,u)),y),z))).
1055 [hyper,1,539,4] P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))).
1056 [hyper,1,539,2] P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))).
1095 [hyper,1,549,58] P(i(i(x,i(i(i(i(y,z),u),i(i(v,z),u)),w)),i(i(y,v),i(x,w)))).
1252 [hyper,1,68,607] P(i(x,i(i(i(y,i(z,u)),v),i(i(y,i(x,u)),v)))).
2060 [hyper,1,1056,613] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
2119 [hyper,1,68,1095] P(i(i(x,i(i(i(i(y,z),u),i(i(v,z),u)),w)),i(i(i(x,w),v6),i(i(y,v),v6)))).
2456 [hyper,1,253,1252] P(i(i(i(x,i(y,z)),u),i(v,i(i(x,i(v,z)),u)))).
4360 [hyper,1,35,2060] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)).
4362 [hyper,1,27,2060] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),v6),v7),i(i(w,v6),v7))).
5051 [hyper,1,2456,2119] P(i(x,i(i(y,i(x,z)),i(i(i(y,z),u),i(i(v,w),u))))).
8645 [hyper,1,1055,4360] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8)).
8655 [hyper,1,18,4362] P(i(i(x,i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6))),i(i(v6,v7),i(x,v7)))).
10242 [hyper,1,5051,5] P(i(i(x,i(i(i(n(y),n(z)),i(z,y)),u)),i(i(i(x,u),v),i(i(w,v6),v)))).
17518 [hyper,1,35,8645] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8),v9),v9)).
45108 [hyper,1,10242,17518] P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),

i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),i(i(i(n(v9),n(v10)),i(v10,v9)),v11)),v11),v12),i(i(v13,v14),v12))).
121133 [hyper,1,8655,45108] P(i(i(x,y),i(i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),
i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),x)),y))).
287254 [hyper,1,830,121133] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),
i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),
i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)).

----> UNIT CONFLICT at 85925.83 sec ----> 287296 [binary,287295.1,6.1] $ANS(REZMV).

Length of proof is 35. Level of proof is 13.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(x,i(y,x))).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(i(x,y),y),i(i(y,x),x))).
5 [] P(i(i(n(x),n(y)),i(y,x))).
6 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(n(b11),n(b12)),i(b12,b11)),b13)),b13),
i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZMV).
18 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
19 [hyper,1,2,3] P(i(x,i(i(y,z),i(i(z,u),i(y,u))))).
20 [hyper,1,3,2] P(i(i(i(x,y),z),i(y,z))).
27 [hyper,1,18,3] P(i(i(x,y),i(i(i(x,z),u),i(i(y,z),u)))).
29 [hyper,1,4,19] P(i(i(i(i(x,y),i(i(y,z),i(x,z))),u),u)).
35 [hyper,1,20,4] P(i(x,i(i(x,y),y))).
36 [hyper,1,20,3] P(i(x,i(i(x,y),i(z,y)))).
58 [hyper,1,3,27] P(i(i(i(i(i(x,y),z),i(i(u,y),z)),v),i(i(x,u),v))).
68 [hyper,1,18,29] P(i(i(x,i(y,z)),i(x,i(i(z,u),i(y,u))))).
88 [hyper,1,3,35] P(i(i(i(i(x,y),y),z),i(x,z))).
91 [hyper,1,3,36] P(i(i(i(i(x,y),i(z,y)),u),i(x,u))).
253 [hyper,1,18,88] P(i(i(x,i(y,z)),i(y,i(x,z)))).
260 [hyper,1,88,18] P(i(i(x,y),i(i(z,x),i(z,y)))).
539 [hyper,1,253,253] P(i(x,i(i(y,i(x,z)),i(y,z)))).
549 [hyper,1,3,253] P(i(i(i(x,i(y,z)),u),i(i(y,i(x,z)),u))).
603 [hyper,1,260,260] P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))).
607 [hyper,1,91,260] P(i(x,i(i(y,i(x,z)),i(y,i(u,z))))).
613 [hyper,1,35,260] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)).
1055 [hyper,1,539,4] P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))).
1056 [hyper,1,539,2] P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))).
1095 [hyper,1,549,58] P(i(i(x,i(i(i(i(y,z),u),i(i(v,z),u)),w)),i(i(y,v),i(x,w)))).
1252 [hyper,1,68,607] P(i(x,i(i(i(y,i(z,u)),v),i(i(y,i(x,u)),v)))).
2060 [hyper,1,1056,613] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
2119 [hyper,1,68,1095] P(i(i(x,i(i(i(i(y,z),u),i(i(v,z),u)),w)),i(i(i(x,w),v6),i(i(y,v),v6)))).
2456 [hyper,1,253,1252] P(i(i(i(x,i(y,z)),u),i(v,i(i(x,i(v,z)),u)))).
4351 [hyper,1,260,2060] P(i(i(x,i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6))),i(x,v6))).
4360 [hyper,1,35,2060] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)).
5051 [hyper,1,2456,2119] P(i(x,i(i(y,i(x,z)),i(i(i(y,z),u),i(i(v,w),u))))).
8596 [hyper,1,603,4351] P(i(i(x,i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6))),i(i(v7,x),i(v7,v6)))).
8645 [hyper,1,1055,4360] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8)).

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10242 [hyper,1,5051,5] P(i(i(x,i(i(i(n(y),n(z)),i(z,y)),u)),i(i(i(x,u),v),i(i(w,v6),v)))).
17518 [hyper,1,35,8645] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),
  i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),v9),v9)).
45108 [hyper,1,10242,17518] P(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),
  i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),i(i(i(n(v9),n(v10)),i(v10,v9)),v11)),v11),v12),
  i(i(v13,v14),v12))).
121135 [hyper,1,8596,45108] P(i(i(x,i(i(i(i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6),
  i(i(i(i(v7,v8),v8),i(i(v8,v7),v7)),v9)),v9),i(i(i(n(v10),n(v11)),i(v11,v10)),v12)),v12),
  i(i(v13,i(v14,v13)),v15))),i(x,v15))).
287295 [hyper,1,1056,121135] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),
  i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),
  i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)).
```

For the other so-called half, that using as hypothesis the Rezus-style single axiom for the full *MV*, I conjectured that a much larger value for max_weight might be needed. My reason for this view rests with the possibility that, at least in the beginning, long formulas might be needed in that the Rezus formula is so complex. I did abandon the use of ancestor subsumption and submitted the following input file.

An Input File for Deriving a 4-Basis for Full MV from a Rezus Single Axiom

```
% Trying for a proof of MV5 from MV1-4 avoiding 2.22 3.5 3.51.
set(hyper_res).
assign(max_weight,88).
% assign(change_limit_after,2000).
% assign(new_max_weight,24).
assign(max_proofs,-1).
clear(print_kept).
% set(process_input).
set(ancestor_subsume).
set(back_sub).
clear(print_back_sub).
% assign(max_distinct_vars,4).
assign(pick_given_ratio,1).
% assign(max_seconds,40).
assign(max_mem,480000).
assign(report,3600).
set(order_history).
set(input_sos_first).
assign(heat,0).
% assign(dynamic_heat_weight,0).

weight_list(pick_and_purge).
% Following 24 are subformulas, from Overbeek, of a Rezus-style formula, from Ulrich,
%single axiom for all of MV.
weight(i(v,u),2).
weight(n(v11),2).
weight(i(x,i(y,x)),2).
weight(i(i(v8,v9),v9),2).
weight(i(i(v,z),i(v,u)),2).
weight(i(n(v11),n(v12)),2).
weight(i(v14,i(v15,v14)),2).
```

```
weight(i(i(w,i(v6,w)),v7),2).
weight(i(i(v14,i(v15,v14)),v16),2).
weight(i(i(z,u),i(i(v,z),i(v,u))),2).
weight(i(i(n(v11),n(v12)),i(v12,v11)),2).
weight(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),2).
weight(i(i(i(n(v11),n(v12)),i(v12,v11)),v13),2).
weight(i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10),2).
weight(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),2).
weight(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),2).
weight(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),2).
weight(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),2).
weight(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),2).
weight(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),2).
weight(i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16)),2).
weight(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),2).
weight(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16),2).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)),2).
end_of_list.

list(usable).
% condensed detachment
-P(i(x,y)) | -P(x) | P(y).
-P(i(a,i(b,a))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(i(a,b),b),i(i(b,a),a))) | -P(i(i(n(a),n(b)),i(b,a))) |
  $ANS(MVALL).
end_of_list.

list(sos).
% Following is from Ulrich, a Rezus-style single axiom for all of MV.
P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)).
end_of_list.

list(passive).
% Just the implicational Axioms: MV1 - MV3
-P(i(a,i(b,a))) | $ANS(MV1).
-P(i(i(a,b),i(i(b,c),i(a,c)))) | $ANS(MV2).
-P(i(i(i(a,b),b),i(i(b,a),a))) | $ANS(MV3).
% Following is MV4
-P(i(i(n(a),n(b)),i(b,a))) | $ANS(MV4).
end_of_list.

list(demodulators).
(n(n(n(x))) = junk).
% (n(n(x)) = junk).
% (i(i(x,x),y) = junk).
% (i(y,i(x,x)) = junk).
% (i(n(i(x,x)),y) = junk).
```

```
% (i(y,n(i(x,x))) = junk).
(i(x,junk) = junk).
(i(junk,x) = junk).
(n(junk) = junk).
(P(junk) = $T).
end_of_list.
```

Again, I stress the use in list(usable), of the negation of the join. The intent is to obtain a proof of the join of the for targets, a proof with no duplicate steps in it.

A Proof Deriving a 4-Basis for Full MV from a Rezus single Axiom

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on elephant.mcs.anl.gov,
Tue Jun 17 06:45:17 2008
The command was "otter". The process ID is 27338.

-----> EMPTY CLAUSE at 1092.48 sec ----> 171846 [hyper,2,14,170631,51,42] $ANS(MVALL).

Length of proof is 18. Level of proof is 10.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] -P(i(a,i(b,a))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(i(a,b),b),i(i(b,a),a))) | -P(i(i(n(a),n(b)),i(b,a))) |
  $ANS(MVALL).
3 [] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)).
13 [hyper,1,3,3] P(i(x,i(y,i(z,y)))).
14 [hyper,1,13,13] P(i(x,i(y,x))).
16 [hyper,1,3,13] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),i(i(i(n(v9),n(v10)),i(v10,v9)),v11)),v11)).
20 [hyper,1,16,14] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8)).
21 [hyper,1,16,13] P(i(x,i(i(n(y),n(z)),i(z,y)))).
24 [hyper,1,20,14] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
25 [hyper,1,20,13] P(i(x,i(i(i(y,z),z),i(i(z,y),y)))).
29 [hyper,1,24,14] P(i(i(x,y),i(i(z,x),i(z,y)))).
31 [hyper,1,29,29] P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))).
36 [hyper,1,29,24] P(i(i(x,i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6))),i(x,v6))).
42 [hyper,1,21,21] P(i(i(n(x),n(y)),i(y,x))).
51 [hyper,1,25,25] P(i(i(i(x,y),y),i(i(y,x),x))).
55 [hyper,1,51,25] P(i(i(i(i(i(x,y),y),i(i(y,x),x)),z),z)).
95 [hyper,1,24,31] P(i(i(x,i(y,z)),i(x,i(y,i(u,z))))).
115 [hyper,1,24,55] P(i(x,i(i(x,y),y))).
131 [hyper,1,31,115] P(i(x,i(i(y,i(x,z)),i(y,z)))).
1923 [hyper,1,95,131] P(i(x,i(i(y,i(x,z)),i(u,i(y,z))))).
170631 [hyper,1,36,1923] P(i(i(x,y),i(i(y,z),i(x,z)))).
```

The following aside might indeed prove useful, especially in the context of proof shortening. The presence in an input file, designed to prove a conjunction of two or more formulas or equations, of the negation of each member (in the passive list) can have value in addition to that of monitoring. Specifically, success with such an input file will include, in addition to a proof of the conjunction, proofs of its members. A glance at those so-called subproofs suggests where improvements in proof length might be made. For example, you could turn to *cramming*, a strategy I introduced to be used mainly for proof shortening; for some discussion, see the notebook titled *The Joy of Solving a Decades-Old Mystery: Success with the BCK and BCI Logics*. With this strategy, usually with level saturation, you place the proof steps of, say, the longest subproof in list(sos) with the intention of forcing them to do double duty, triple duty, or more. In other words, you are trying to force the various subproofs to have larger intersections than they did in the original success.

Well, the experiment that has been running, while I have been focusing on three proofs found in the presence of ancestor subsumption, completed. In addition to avoiding the use of McCune's ancestor subsumption, I also switched from a level-saturation approach, set(sos_queue), to a complexity-preference approach with a value of 4 assigned to the pick_given_ratio. Whereas the first of the three cited proofs required just under 86,000 CPU-seconds and a retention of clause (287234), the just-completed experiment required just under 157,000 CPU-seconds and retention of clause (203736). The first of the three proofs (found in a single run through the use of ancestor subsumption) has length 41; in contrast, the only proof obtained in the just-completed experiment has length 34. Because of the use of the ordinarily expensive procedure of ancestor subsumption, I did expect to get a proof sooner than was found. Instead, far more CPU time was spent, the cause apparently being a complexity-preference search in place of a breadth-first (level-saturation) search. So much for conducting an experiment in which two changes were made. For one additional bit of data, when I tried again without ancestor subsumption and with a level-saturation approach, OTTER found a 65-step level-12 proof in just over 132,000 CPU-seconds and with retention of clause (160070). This data may in some oblique way assist you in meeting one of the challenges I offer in this or other notebooks.

6. Proof Shortening

In this section, I give proofs that result from my various studies of proofs that you have seen earlier in this notebook. I also discuss some of the approaches that can be used to find shorter proofs, approaches I employed. Before giving the proofs and approaches, I offer the following bit of history that nicely illustrates an unusual phenomenon.

When seeking a short proof, occasionally a delightful discovery is made. Specifically, sometimes the proof in hand has length, say, 22, and the proof that is discovered has length 21 with the property that all of the deduced steps of the shorter proof are among the deduced steps of the longer. Of course, the history of the deduced steps in the newer and shorter proof is different from that in the longer proof. If your goal were to find such a pair, you might proceed in the following manner.

Specifically, you could take the input file I shall soon give and conduct two experiments. In the first, you retain the comment (%) in front of a demodulator; in the second, you remove the comment to permit the demodulator to be used. In the first experiment, you will find in the output a 22-step proof of a system that I formerly attributed to Church, but that should be attributed to Lukasiewicz. In the second experiment, you will find in the output a 21-step proof of the same system—a proof with a most charming property. Indeed, all of the deduced steps of the 21-step proof are among the 22 deduced steps of the longer proof, that yielded in the first experiment.

If you enjoy puzzles, I invite you to decide about the origin of the demodulator in focus. Then—and pause for some comments before giving answers—after I provide the two proofs in question, you might enjoy an analysis designed to explain how this phenomenon occurred, a so-called nested pair of proofs.

An Input File for Finding a Nested Pair of Proofs

```
% Trying to proof check the first 21-step proof of Church I found,
% from Chapter 13 of my fourth book.
set(hyper_res).
assign(max_weight,2).
assign(max_proofs,-1).
clear(print_kept).
```

```
set(back_sub).
clear(print_back_sub).
assign(pick_given_ratio,4).
assign(max_mem,480000).
assign(max_seconds,2).
assign(report,3600).
set(order_history).
set(input_sos_first).
assign(heat,0).
assign(dynamic_heat_weight,0).

weight_list(pick_and_purge).
% Following 22 yield a 22-step proof of an axiom system of Lukasiewicz.
weight(P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))),2).
weight(P(i(i(x,y),i(i(n(x),x),y))),2).
weight(P(i(i(i(n(x),y),z),i(x,z))),2).
weight(P(i(n(i(i(n(x),x),x)),y)),2).
weight(P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))),2).
weight(P(i(i(x,y),i(i(n(i(y,z)),i(y,z)),i(x,z)))),2).
weight(P(i(i(x,y),i(n(i(i(n(z),z),z)),y))),2).
weight(P(i(i(x,i(n(i(y,z)),i(y,z))),i(i(u,y),i(x,i(u,z))))),2).
weight(P(i(x,i(n(i(i(n(y),y),y)),z))),2).
weight(P(i(i(x,i(n(y),y)),i(z,i(x,y)))),2).
weight(P(i(i(n(x),y),i(z,i(i(y,x),x)))),2).
weight(P(i(i(x,i(y,z)),i(i(n(z),y),i(x,z)))),2).
weight(P(i(i(x,i(n(y),z)),i(i(u,i(z,y)),i(x,i(u,y))))),2).
weight(P(i(i(n(x),n(y)),i(y,x))),2).
weight(P(i(x,i(y,x))),2).
weight(P(i(i(i(x,y),z),i(y,z))),2).
weight(P(i(n(x),i(x,y))),2).
weight(P(i(i(n(x),y),i(n(y),x))),2).
weight(P(i(n(i(x,y)),x)),2).
weight(P(i(i(x,y),i(n(i(x,z)),y))),2).
weight(P(i(i(x,i(y,i(z,u))),i(i(z,y),i(x,i(z,u))))),2).
weight(P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))),2).
end_of_list.

list(usable).
% condensed detachment
-P(i(x,y)) | -P(x) | P(y).
% The following disjunction is the negation of the Church axiom system.
-P(i(q,i(p,q))) | -P(i(i(p,i(q,r)),i(i(p,q),i(p,r)))) | -P(i(i(n(p),n(q)),i(q,p))) |
$ANS(Church_18_35_49).
end_of_list.

list(sos).
% Luka 1 2 3.
P(i(i(x,y),i(i(y,z),i(x,z)))).
P(i(i(n(x),x),x)).
P(i(x,i(n(x),y))).
end_of_list.

list(passive).
```

```
% Following are negations of the members of the Church system.
-P(i(q,i(p,q))) | $ANS(step_18).
-P(i(i(p,i(q,r)),i(i(p,q),i(p,r)))) | $ANS(step_35).
-P(i(i(n(p),n(q)),i(q,p))) | $ANS(step_49).
end_of_list.

list(demodulators).
% (P(i(i(x,y),i(i(n(x),x),y))) = junk).
% (n(n(n(x))) = junk).
% % (n(n(x)) = junk).
% % (i(i(x,x),y) = junk).
% % (i(y,i(x,x)) = junk).
% % (i(n(i(x,x)),y) = junk).
% % (i(y,n(i(x,x))) = junk).
% (i(x,junk) = junk).
% (i(junk,x) = junk).
% (n(junk) = junk).
% (P(junk) = $T).
end_of_list.

list(hot).
-P(i(x,y)) | -P(x) | P(y).
% Luka 1 2 3.
P(i(i(x,y),i(i(y,z),i(x,z)))).
P(i(i(n(x),x),x)).
P(i(x,i(n(x),y))).
end_of_list.
```

First Proof of a Nested Pair

```
-----> EMPTY CLAUSE at   0.02 sec ----> 58 [hyper,2,34,56,30] $ANS(Church_18_35_49).

Length of proof is 22.  Level of proof is 15.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] -P(i(q,i(p,q))) | -P(i(i(p,i(q,r)),i(i(p,q),i(p,r)))) | -P(i(i(n(p),n(q)),i(q,p))) | $ANS(Church_18_35_49).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(n(x),x),x)).
5 [] P(i(x,i(n(x),y))).
13 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
14 [hyper,1,3,4] P(i(i(x,y),i(i(n(x),x),y))).
15 [hyper,1,3,5] P(i(i(i(n(x),y),z),i(x,z))).
16 [hyper,1,5,4] P(i(n(i(i(n(x),x),x)),y)).
17 [hyper,1,13,13] P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))).
18 [hyper,1,13,14] P(i(i(x,y),i(i(n(i(y,z)),i(y,z)),i(x,z)))).
20 [hyper,1,3,16] P(i(i(x,y),i(n(i(i(n(z),z),z)),y))).
21 [hyper,1,17,18] P(i(i(x,i(n(i(y,z)),i(y,z))),i(i(u,y),i(x,i(u,z))))).
22 [hyper,1,15,20] P(i(x,i(n(i(i(n(y),y),y)),z))).
23 [hyper,1,21,22] P(i(i(x,i(n(y),y)),i(z,i(x,y)))).
24 [hyper,1,13,23] P(i(i(n(x),y),i(z,i(i(y,x),x)))).
```

26 [hyper,1,21,24] P(i(i(x,i(y,z)),i(i(n(z),y),i(x,z)))).
29 [hyper,1,17,26] P(i(i(x,i(n(y),z)),i(i(u,i(z,y)),i(x,i(u,y))))).
30 [hyper,1,26,5] P(i(i(n(x),n(y)),i(y,x))).
34 [hyper,1,15,30] P(i(x,i(y,x))).
38 [hyper,1,3,34] P(i(i(i(x,y),z),i(y,z))).
41 [hyper,1,38,30] P(i(n(x),i(x,y))).
43 [hyper,1,26,41] P(i(i(n(x),y),i(n(y),x))).
46 [hyper,1,43,41] P(i(n(i(x,y)),x)).
51 [hyper,1,3,46] P(i(i(x,y),i(n(i(x,z)),y))).
53 [hyper,1,29,51] P(i(i(x,i(y,i(z,u))),i(i(z,y),i(x,i(z,u))))).
56 [hyper,1,53,3] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).

Second Proof of a Nested Pair

-----> EMPTY CLAUSE at 0.01 sec ----> 53 [hyper,2,31,49,29] $ANS(Church_18_35_49).

Length of proof is 21. Level of proof is 16.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] -P(i(q,i(p,q))) | -P(i(i(p,i(q,r)),i(i(p,q),i(p,r)))) | -P(i(i(n(p),n(q)),i(q,p))) | $ANS(Church_18_35_49).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(n(x),x),x)).
5 [] P(i(x,i(n(x),y))).
14 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
16 [hyper,1,3,5] P(i(i(i(n(x),y),z),i(x,z))).
17 [hyper,1,5,4] P(i(n(i(i(n(x),x),x)),y)).
18 [hyper,1,14,14] P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))).
19 [hyper,1,3,17] P(i(i(x,y),i(n(i(i(n(z),z),z)),y))).
20 [hyper,1,18,4] P(i(i(x,y),i(i(n(i(y,z)),i(y,z)),i(x,z)))).
21 [hyper,1,16,19] P(i(x,i(n(i(i(n(y),y),y)),z))).
22 [hyper,1,18,20] P(i(i(x,i(n(i(y,z)),i(y,z))),i(i(u,y),i(x,i(u,z))))).
23 [hyper,1,22,21] P(i(i(x,i(n(y),y)),i(z,i(x,y)))).
24 [hyper,1,14,23] P(i(i(n(x),y),i(z,i(i(y,x),x)))).
26 [hyper,1,22,24] P(i(i(x,i(y,z)),i(i(n(z),y),i(x,z)))).
28 [hyper,1,18,26] P(i(i(x,i(n(y),z)),i(i(u,i(z,y)),i(x,i(u,y))))).
29 [hyper,1,26,5] P(i(i(n(x),n(y)),i(y,x))).
31 [hyper,1,16,29] P(i(x,i(y,x))).
33 [hyper,1,3,31] P(i(i(i(x,y),z),i(y,z))).
34 [hyper,1,33,29] P(i(n(x),i(x,y))).
35 [hyper,1,26,34] P(i(i(n(x),y),i(n(y),x))).
37 [hyper,1,35,34] P(i(n(i(x,y)),x)).
42 [hyper,1,3,37] P(i(i(x,y),i(n(i(x,z)),y))).
45 [hyper,1,28,42] P(i(i(x,i(y,i(z,u))),i(i(z,y),i(x,i(z,u))))).
49 [hyper,1,45,3] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).

Before giving the answers to the two puzzles, I shall explain a bit about what you can do when seeking a short proof in the presence of a proof you already possess, about the approach that was just exhibited. A most efficient approach begins by placing the proof steps of the proof you have in hand with corresponding weight templates in weight_list(pick_and_purge), as seen in the preceding input file. You then make a series of runs designed to block, one at a time, each of the known proof steps. You can, as in the given input file, block the use of a formula with demodulation. You are not required, as was the case in the given input file, to constrain the program by assigning a small value to max_weight. Instead, you can assign a value that permits the program to complete a proof that includes one or more formulas that are not among the proof steps used in the beginning. (For those who enjoy history, I found the given 21-step proof after I had searched unsuccessfully for quite a while for a proof of length strictly less than 22; the 21-step proof was found when I first tested McCune's incorporation into OTTER of the hot list strategy.) Now, before discussing the shorter proofs I found, I shall answer the two puzzles.

As for the answer to the first puzzle, that concerning the origin of the key demodulator found in the given input file, the demodulator corresponds precisely to the second deduced step of the 22-step proof. The inclusion of this demodulator, to block retention of the second step (and the formulas it subsumes), forces the program to seek another path for proving the so-called Church system. As for the second puzzle, a review of the two given proofs shows that the sixth deduced step, although the same in both proofs, has different parents. In the longer 22-step proof, one of the parents of the sixth step is the second deduced step. Its absence by design (with the use of demodulation blocking) in the second 21-step proof resulted in the program successfully finding a different set of parents for the sixth step. Such phenomena add greatly to my pleasure.

And now the time has come for the presentation of shorter proofs germane to *MV*-implicational and to *MV*-full. I shall offer them in an order that mirrors the studies as discussed in this notebook. Therefore, first in focus is a proof that derives the earlier-given Rezus-style formula for the implicational fragment of the Lukasiewicz infinite-valued sentential calculus from the 4-basis consisting of *MV*1, *MV*2, *MV*3, and *MV*5. When the corresponding theorem was studied with the use solely of subformulas, as noted, a 73-step proof was eventually completed. When I sought a shorter proof, with the goal of finding one much shorter, iteration and heavy use of demodulation blocking of various proof steps finally led to the discovery of an 18-step proof. The iteration occurred as I replaced one shorter proof after another still shorter. In the following input file, you will find in list(sos) the 4-basis and in list(passive) the negation of the target Rezus formula. In list(demodulators), you will find those items that are not commented out and that were used to (in effect) prevent certain formulas from participating in the search for a shorter proof. The first two demodulators correspond to two steps that, when blocked, led to progress; the other demodulators, not commented out, propagate "junk" to the point that a discard occurs. I note that other demodulators were almost certainly used on the way from the 73-step proof to the 18-step proof as I focused on one proof after another, always focusing on a strictly shorter proof than I had in the preceding set of experiments. I shall then give the resulting proof.

An Input File Yielding a Short Proof Deriving Rezus from a 4-Basis

```
set(hyper_res).
assign(max_mem,880000).
% assign(max_seconds,2).
% set(sos_queue).
set(input_sos_first).
assign(max_weight,18).
% assign(change_limit_after,200).
% assign(new_max_weight,8).
assign(max_proofs,-1).
assign(max_distinct_vars,18).
assign(pick_given_ratio,2).
assign(bsub_hint_wt,1).
assign(report,5400).
set(keep_hint_subsumers).
% set(keep_hint_equivalents).
set(ancestor_subsume).
set(back_sub).
set(order_history).
% set(process_input).
```

```
clear(print_kept).

weight_list(pick_and_purge).
% Following 19/15 prove Rezus-style for MV imp, temp.ulrich.rezus.mv.out1l6.
weight(P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))),1).
weight(P(i(i(i(x,y),z),i(y,z))),1).
weight(P(i(x,i(i(x,y),y))),1).
weight(P(i(i(i(i(x,y),y),z),i(x,z))),1).
weight(P(i(i(x,y),i(i(z,x),i(z,y)))),1).
weight(P(i(x,i(i(y,i(x,z)),i(y,z)))),1).
weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)),1).
weight(P(i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),i(x,u))),1).
weight(P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))),1).
weight(P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))),1).
weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)),1).
weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)),1).
weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8)),1).
weight(P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),v9),v9)),1).
weight(P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),
  i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)),1).
weight(P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),
  i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),v12),v12)),1).
weight(P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),
  i(i(v12,i(v13,v12)),v14)),v14)),1).
weight(P(i(i(x,i(i(i(i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6),i(i(i(i(v7,v8),v8),
  i(i(v8,v7),v7)),v9)),v9),i(i(i(i(v10,v11),i(v11,v10)),i(v11,v10)),v12)),v12),
  i(i(v13,i(v14,v13)),v15))),i(x,v15))),1).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
  i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),
  i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)),1).
% Following 21 prove the Rezus, shortest as of mid-05-24-08, perhaps temp.ulrich.rezus.mv.out1m3
% weight(P(i(x,i(y,i(z,y)))),1).
% weight(P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))),1).
% weight(P(i(i(i(x,y),z),i(y,z))),1).
% weight(P(i(i(i(x,i(y,x)),z),z)),1).
% weight(P(i(x,i(i(x,y),y))),1).
% weight(P(i(i(x,y),i(i(i(z,i(u,z)),x),y))),1).
% weight(P(i(i(i(i(x,y),y),z),i(x,z))),1).
% weight(P(i(i(x,y),i(i(z,x),i(z,y)))),1).
% weight(P(i(x,i(i(y,i(x,z)),i(y,z)))),1).
% weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)),1).
% weight(P(i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),i(x,u))),1).
% weight(P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))),1).
% weight(P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))),1).
% weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)),1).
% weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)),1).
% weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8)),1).
% weight(P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),v9),v9)),1).
% weight(P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
```

```
  i(i(v7,v6),v6)),v8)),v8),i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)),1).
% weight(P(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),v12),v12)),1).
% weight(P(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7), r
  i(i(v7,v6),v6)),v8)),v8),i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),
  i(i(v12,i(v13,v12)),v14)),v14)),1).
% weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
  i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),
  i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)),1).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
  i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16)),2).  % Rezus-style axiom for MV
end_of_list.

list(usable).
-P(i(x,y)) | -P(x) | P(y).
% -P(i(a,a)) | -P(i(i(a,i(b,c)),i(b,i(a,c)))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(a,i(a,b)),i(a,b))) |
  $ANS(all).
end_of_list.

list(sos).
% Following 4-basis, used by Veroff, for MV implicational fragment.
P(i(x,i(y,x))).
P(i(i(x,y),i(i(y,z),i(x,z)))).
P(i(i(i(x,y),y),i(i(y,x),x))).
P(i(i(i(x,y),i(y,x)),i(y,x))).
end_of_list.

list(passive).
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZPROOF).
end_of_list.

list(demodulators).
(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)) = junk).
(P(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),
  i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),v12),v12)) = junk).
% (i(i(x,x),y) = junk).
% (i(y,i(x,x)) = junk).
(i(x,junk) = junk).
(i(junk,x) = junk).
(P(junk) = $T).
end_of_list.
```

An 18-Step Proof Deriving Rezus from a 4-Basis in Implicational MV

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on octopus.mcs.anl.gov,
Mon May 26 08:31:35 2008
The command was "otter".  The process ID is 25900.

----> UNIT CONFLICT at   0.78 sec ----> 2480 [binary,2479.1,7.1] $ANS(REZPROOF).
```

Length of proof is 18. Level of proof is 15.

```
---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(x,i(y,x))).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(i(x,y),y),i(i(y,x),x))).
5 [] P(i(i(i(x,y),i(y,x)),i(y,x))).
7 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZPROOF).
14 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
16 [hyper,1,3,2] P(i(i(i(x,y),z),i(y,z))).
29 [hyper,1,16,4] P(i(x,i(i(x,y),y))).
38 [hyper,1,3,29] P(i(i(i(i(x,y),y),z),i(x,z))).
51 [hyper,1,38,14] P(i(i(x,y),i(i(z,x),i(z,y)))).
59 [hyper,1,38,51] P(i(x,i(i(y,i(x,z)),i(y,z)))).
92 [hyper,1,59,5] P(i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),i(x,u))).
93 [hyper,1,59,4] P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))).
94 [hyper,1,59,2] P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))).
132 [hyper,1,38,94] P(i(x,i(i(x,i(i(y,i(z,y)),u)),u))).
2360 [hyper,1,132,51] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
2372 [hyper,1,29,2360] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)).
2383 [hyper,1,93,2372] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),i
  (i(v7,v6),v6)),v8)),v8)).
2390 [hyper,1,29,2383] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),v9),v9)).
2408 [hyper,1,92,2390] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)).
2417 [hyper,1,132,2408] P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),i(i(v12,i(v13,v12)),v14)),v14)).
2430 [hyper,1,51,2417] P(i(i(x,i(i(i(i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6),
  i(i(i(i(v7,v8),v8),i(i(v8,v7),v7)),v9)),v9),i(i(i(i(v10,v11),i(v11,v10)),
  i(v11,v10)),v12)),v12),i(i(v13,i(v14,v13)),v15))),i(x,v15))).
2479 [hyper,1,94,2430] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
  i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16)).
```

I know of no shorter proof for this theorem; however, a reduction from 73 to 18 was quite pleasing. If you find a shorter proof, I would enjoy receiving a copy of it by e-mail.

Next is my study designed to find a shorter proof, from the Rezus formula, of the 4-basis. In that I had in hand an 18-step proof, I did not expect to make much progress in the context of proof length. I was in that regard not disappointed with the 15-step proof yielded by the following input file; but, of course, I always want a bigger reduction in length than three.

An Input File for Deriving a 4-Basis from the Rezus Formula

```
set(hyper_res).
assign(max_mem,880000).
assign(max_seconds,14).
% set(sos_queue).
set(input_sos_first).
```

```
assign(max_weight,15).
% assign(change_limit_after,50).
% assign(new_max_weight,6).
assign(max_proofs,-1).
% assign(max_distinct_vars,19).
assign(pick_given_ratio,4).
assign(bsub_hint_wt,1).
assign(report,5400).
set(keep_hint_subsumers).
% set(keep_hint_equivalents).
set(ancestor_subsume).
set(back_sub).
set(order_history).
% set(process_input).
clear(print_kept).

weight_list(pick_and_purge).
% Following 16/11 prove the 4-basis, from the Rezus for MV implicational,
% temp.ulrich.rezus.mv.out6b.
weight(P(i(x,i(y,i(z,y)))),-1).
weight(P(i(x,i(y,x))),-1).
weight(P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),
  i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)),-1).
weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8)),-1).
weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)),-1).
weight(P(i(x,i(i(i(y,z),z),i(i(z,y),y)))),-1).
weight(P(i(i(i(x,y),i(y,x)),i(y,x))),-1).
weight(P(i(i(x,y),i(i(z,x),i(z,y)))),-1).
weight(P(i(i(i(x,y),y),i(i(y,x),x))),-1).
weight(P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))),-1).
weight(P(i(i(i(i(i(x,i(y,x)),i(x,z)),i(i(y,x),z)),i(i(y,x),z)),i(x,z))),-1).
weight(P(i(x,i(i(x,y),y))),-1).
weight(P(i(x,i(i(y,i(x,z)),i(y,z)))),-1).
weight(P(i(i(x,i(i(y,i(i(z,i(y,u)),i(z,u))),v)),i(x,v))),-1).
weight(P(i(i(x,i(y,z)),i(y,i(x,z)))),-1).
weight(P(i(i(x,y),i(i(y,z),i(x,z)))),-1).
% Following 15 prove one of the 4 basis elements, with one step not in the 18,
% temp.ulrich.rezus.mv.out6a.
weight(P(i(x,i(y,i(z,y)))),0).
weight(P(i(x,i(y,x))),0).
weight(P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),
  i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)),0).
weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8)),0).
weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)),0).
weight(P(i(x,i(i(i(y,z),z),i(i(z,y),y)))),0).
weight(P(i(i(x,y),i(i(z,x),i(z,y)))),0).
weight(P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))),0).
weight(P(i(i(i(i(i(x,i(y,x)),i(x,z)),i(i(y,x),z)),i(i(y,x),z)),i(x,z))),0).
weight(P(i(i(x,i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6))),i(x,v6))),0).
weight(P(i(i(x,i(y,z)),i(x,i(y,i(u,z))))),0).
weight(P(i(x,i(i(x,y),y))),0).
weight(P(i(x,i(i(y,i(x,z)),i(y,z)))),0).
weight(P(i(x,i(i(y,i(x,z)),i(u,i(y,z))))),0).
```

```
weight(P(i(i(x,y),i(i(y,z),i(x,z)))),0).
% Following 25, including target, are subformulas of the Rezus for the implicational of MV,
% from Ross.
weight(i(v,u),1).
weight(i(x,i(y,x)),1).
weight(i(w,i(v6,w)),1).
weight(i(i(v8,v9),v9),1).
weight(i(i(v9,v8),v8),1).
weight(i(i(v,z),i(v,u)),1).
weight(i(v14,i(v15,v14)),1).
weight(i(i(w,i(v6,w)),v7),1).
weight(i(i(v11,v12),i(v12,v11)),1).
weight(i(i(v14,i(v15,v14)),v16),1).
weight(i(i(z,u),i(i(v,z),i(v,u))),1).
weight(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),1).
weight(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),1).
weight(i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10),1).
weight(i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13),1).
weight(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),1).
weight(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),1).
weight(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),1).
weight(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),1).
weight(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),1).
weight(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),1).
weight(i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16)),1).
weight(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),1).
weight(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16),1).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16)),1).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16)),1).  % Rezus-style axiom for MV
end_of_list.

list(usable).
-P(i(x,y)) | -P(x) | P(y).
-P(i(a,i(b,a))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(i(a,b),b),i(i(b,a),a))) |
  -P(i(i(i(a,b),i(b,a)),i(b,a))) | $ANS(MVBASISALL).
end_of_list.

list(sos).
P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16)).
% Rezus-style axiom for MV
end_of_list.
```

```
list(passive).
% Following 4-basis, used by Veroff, for MV implicational fragment.
-P(i(a,i(b,a))) | $ANS(MVBASIS).
-P(i(i(a,b),i(i(b,c),i(a,c)))) | $ANS(MVBASIS).
-P(i(i(i(a,b),b),i(i(b,a),a))) | $ANS(MVBASIS).
-P(i(i(i(a,b),i(b,a)),i(b,a))) | $ANS(MVBASIS).
% Following is neg of Ulrich's 37-symbol single axiom for the implicational fragment of MV.
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7)),
  i(i(b6,b),b7))) | $ANS(UL37).
% Following is neg of the Rezus-style single axiom for the implicational fragment of MV.
-P(i(i(i(a1,i(i(i(a2,a2),i(i(a3,a3),i(i(a4,a4),i(i(b,b),i(a1,a6))))),a6)),i(i(i(i(b6,b7),i(i(b7,b8),
  i(b6,b8))),i(i(i(i(b9,i(b9,b10)),i(b9,b10)),i(i(i(b11,b12),i(i(b12,b13),
  i(b11,b13))),b14)),b14),b15)),b15),i(i(b16,i(i(i(b17,b17),i(i(b18,b18),i(i(b19,b19),
  i(i(b20,b20),i(b16,b21))))),b21)),b22))),b22)) | $ANS(REZ).
end_of_list.

list(demodulators).
(P(i(x,i(i(i(y,z),i(z,y)),i(z,y)))) = junk).
% (i(i(x,x),y) = junk).
% (i(y,i(x,x)) = junk).
(i(x,junk) = junk).
(i(junk,x) = junk).
(P(junk) = $T).
end_of_list.
```

A Shorter Proof Deriving the 4-Basis from the Rezus Formula

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on jaguar.mcs.anl.gov,
Wed May 28 16:34:24 2008
The command was "otter". The process ID is 27272.

-----> EMPTY CLAUSE at 9.22 sec ----> 14390 [hyper,2,16,14364,33,27] $ANS(MVBASISALL).

Length of proof is 15. Level of proof is 10.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] -P(i(a,i(b,a))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(i(a,b),b),i(i(b,a),a))) | -P(i(i(i(a,b),i(b,a)),i(b,a))) |
  $ANS(MVBASISALL).
3 [] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16)).
15 [hyper,1,3,3] P(i(x,i(y,i(z,y)))).
16 [hyper,1,15,15] P(i(x,i(y,x))).
18 [hyper,1,3,15] P(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)).
22 [hyper,1,18,16] P(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8)).
25 [hyper,1,22,16] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
26 [hyper,1,22,15] P(i(x,i(i(i(y,z),z),i(i(z,y),y)))).
27 [hyper,1,22,3] P(i(i(i(x,y),i(y,x)),i(y,x))).
```

32 [hyper,1,25,16] P(i(i(x,y),i(i(z,x),i(z,y)))).
33 [hyper,1,26,26] P(i(i(i(x,y),y),i(i(y,x),x))).
36 [hyper,1,32,32] P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))).
38 [hyper,1,22,32] P(i(i(i(i(i(x,i(y,x)),i(x,z)),i(i(y,x),z)),i(i(y,x),z)),i(x,z))).
71 [hyper,1,38,26] P(i(x,i(i(x,y),y))).
75 [hyper,1,36,71] P(i(x,i(i(y,i(x,z)),i(y,z)))).
121 [hyper,1,75,32] P(i(i(x,i(i(i(y,z),i(i(u,y),i(u,z))),v)),i(x,v))).
14364 [hyper,1,121,75] P(i(i(x,y),i(i(y,z),i(x,z)))).

As you might expect after glancing at the four basis elements, the member of the basis that would appear to be the most difficult to prove was proved last. Also, if you glance at the input file, you will see sets of *resonators*, weight templates that contain a formula and a value that indicates its relative importance. The smaller the value, the greater is the importance. A resonator is a formula or equation whose function pattern or shape is what is important; all variables are treated in such as indistinguishable. If a deduced item matches a resonator, except for possibly different variables, it is assigned the value that is assigned to the corresponding resonator. You probably have guessed that the inclusion of a set of resonators corresponding to a subproof, of one of the members of the 4-basis, suggests another avenue that often merits consideration. That avenue has you take the steps of the chosen subproof, place them in list(sos), include the command set(sos_first), and see what happens, with the goal of proving the join. If all goes according to plan, as it does sometimes, a shorter proof of the entire set of members will be produced because of (in effect) forcing the steps of the chosen subproof into subproofs of the remaining members. Yes, one or more of the new subproofs may indeed be longer than the ones you had; but the proof of the entire set, because of sharing various proof steps, may be shorter.

For the next proof of interest, the focus is on the Ulrich 37-symbol single axiom and its derivation from the 4-basis. The goal was to find a proof shorter than the 13-step proof offered earlier in this notebook. Now, in place of another input file, I shall leave it to you (if you wish to tackle the problem) the production of an input file that yields the following 11-step proof, the shortest known to me at this time.

A Short Proof Deriving the Ulrich 37-Symbol Single Axiom from the 4-Basis

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on octopus.mcs.anl.gov,
Tue May 27 16:42:19 2008
The command was "otter".  The process ID is 359.

----> UNIT CONFLICT at   0.77 sec ----> 5735 [binary,5733.1,17.1] $ANS(UL).

Length of proof is 11.  Level of proof is 8.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(x,i(y,x))).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(i(x,y),y),i(i(y,x),x))).
5 [] P(i(i(i(x,y),i(y,x)),i(y,x))).
17 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7)),
  i(i(b6,b),b7))) | $ANS(UL).
23 [hyper,1,2,2] P(i(x,i(y,i(z,y)))).
24 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
31 [hyper,1,4,23] P(i(i(i(x,i(y,x)),z),z)).
34 [hyper,1,24,24] P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))).
70 [hyper,1,34,31] P(i(i(x,y),i(i(i(z,i(u,z)),i(y,v)),i(x,v)))).
```

76 [hyper,1,34,4] P(i(i(x,i(y,z)),i(i(i(z,y),y),i(x,z)))).
4390 [hyper,1,76,5] P(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x))).
4471 [hyper,1,2,4390] P(i(x,i(i(i(y,z),z),i(i(i(y,z),i(z,y)),y)))).
4740 [hyper,1,4,4471] P(i(i(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x)),z),z)).
5493 [hyper,1,34,4740] P(i(i(x,y),i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(y,v)),i(x,v)))).
5733 [hyper,1,70,5493] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),
i(v6,w)),v7)),i(i(v6,v),v7))).

I will provide the following small hints. First, ancestor subsumption was employed. Second, some iteration occurred, moving from the 13-step proof to a 12-step and so on. One demodulator is present in the input file I used.

For the last item in this so-called grouping, before turning to other bases, if available, a short proof that derives the 4-basis from the Ulrich axiom is in order. Again, I leave to you the (perhaps) pleasure of producing the appropriate input file. I do offer you the following proof.

A Short Proof Deriving the 4-Basis from the Ulrich Axiom

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on octopus.mcs.anl.gov,
Fri May 30 18:51:47 2008
The command was "otter". The process ID is 10641.

-----> EMPTY CLAUSE at   0.34 sec ----> 1783 [hyper,2,16,36,1748,77] $ANS(MVBASISALL).

Length of proof is 15. Level of proof is 12.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] -P(i(a,i(b,a))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(i(a,b),b),i(i(b,a),a))) | -P(i(i(i(a,b),i(b,a)),i(b,a))) |
  $ANS(MVBASISALL).
3 [] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),i(v6,w)),v7)),i(i(v6,v),v7))).
15 [hyper,1,3,3] P(i(i(x,y),i(z,i(u,z)))).
16 [hyper,1,15,15] P(i(x,i(y,x))).
18 [hyper,1,3,15] P(i(i(x,y),i(z,i(i(i(i(u,v),v),i(i(i(u,v),i(v,u)),u)),i(y,w)),i(x,w))))).
24 [hyper,1,18,16] P(i(x,i(i(i(i(y,z),z),i(i(i(y,z),i(z,y)),y)),i(i(u,v),w)),i(v,w)))).
25 [hyper,1,24,24] P(i(i(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x)),i(i(z,u),v)),i(u,v))).
28 [hyper,1,25,24] P(i(i(i(x,y),z),i(y,z))).
31 [hyper,1,16,28] P(i(x,i(i(i(y,z),u),i(z,u)))).
36 [hyper,1,3,31] P(i(i(x,y),i(i(y,z),i(x,z)))).
45 [hyper,1,36,28] P(i(i(i(x,y),z),i(i(i(u,x),y),z))).
53 [hyper,1,25,45] P(i(x,i(i(i(y,x),i(x,y)),y))).
58 [hyper,1,45,25] P(i(i(i(x,i(i(i(y,z),z),i(i(i(y,z),i(z,y)),y))),i(i(u,v),w)),i(v,w))).
63 [hyper,1,36,53] P(i(i(i(i(i(x,y),i(y,x)),x),z),i(y,z))).
68 [hyper,1,45,63] P(i(i(i(x,i(i(i(y,z),i(z,y)),y)),u),i(z,u))).
77 [hyper,1,63,68] P(i(i(i(x,y),i(y,x)),i(y,x))).
1748 [hyper,1,63,58] P(i(i(i(x,y),y),i(i(y,x),x))).
```

Now the focus switches from single axioms for *MV*-implicational to 3-bases. First under study is the Ulrich 3-basis and an attempt to find a proof shorter than length 54, which is the length of the proof that derived the Rezus formula from the Ulrich 3-basis when the subformula strategy was solely employed. The following long input file illustrates, or at least suggests, how I approached the problem, where iteration was one of the components and demodulation blocking another. You will find in the input file, commented out in some cases, resonators that correspond to proofs found along the way, shorter and still shorter in general. You will also see how demodulation was used, sometimes along the way, to block the use of various formulas from participating in a proof.

```
set(hyper_res).
assign(max_mem,880000).
% assign(max_seconds,2).
% set(sos_queue).
set(input_sos_first).
assign(max_weight,18).
% assign(change_limit_after,100).
% assign(new_max_weight,8).
assign(max_proofs,-1).
assign(max_distinct_vars,17).
assign(pick_given_ratio,4).
assign(bsub_hint_wt,1).
assign(report,5400).
set(keep_hint_subsumers).
% set(keep_hint_equivalents).
set(ancestor_subsume).
set(back_sub).
set(order_history).
% set(process_input).
clear(print_kept).

weight_list(pick_and_purge).
% Following 26/14 has 2 not in preceding 26-step, proof of Rezus, from Ulrich 3-basis,
% temp.ulrich.rezus.mv.ot7i
weight(P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))),2).
weight(P(i(i(i(x,y),z),i(y,z))),2).
weight(P(i(i(i(i(i(x,y),i(y,x)),x),z),i(i(i(x,y),y),z))),2).
weight(P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))),2).
weight(P(i(i(x,y),i(z,i(i(y,u),i(x,u))))),2).
weight(P(i(x,i(i(i(y,x),i(x,y)),y))),2).
weight(P(i(i(x,i(i(y,z),i(z,y))),i(z,i(x,y)))),2).
weight(P(i(i(x,y),i(i(z,x),i(z,y)))),2).
weight(P(i(x,i(i(x,y),y))),2).
weight(P(i(i(x,i(y,z)),i(y,i(x,z)))),2).
weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)),2).
weight(P(i(i(i(x,y),y),i(i(y,x),x))),2).
weight(P(i(x,i(i(y,i(x,z)),i(y,z)))),2).
weight(P(i(i(i(x,y),i(y,x)),i(y,x))),2).
weight(P(i(i(x,i(i(i(y,z),z),i(i(z,y),y))),u)),i(x,u))),2).
weight(P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))),2).
weight(P(i(i(x,i(i(i(y,z),i(z,y)),i(z,y)),u)),i(x,u))),2).
weight(P(i(x,i(i(x,i(i(i(y,z),z),i(i(z,y),y)),u)),u))),2).
weight(P(i(x,i(i(x,i(i(y,i(z,y)),u)),u))),2).
weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)),2).
weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8)),2).
weight(P(i(i(x,i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6),i(i(i(v7,v8),v8),
  i(i(v8,v7),v7)),v9)),v9),v10)),i(x,v10))),2).
weight(P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),
```

i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)),2).
weight(P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),i(i(v12,i(v13,v12)),v14)),v14)),2).
weight(P(i(i(x,i(i(i(i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6),i(i(i(i(v7,v8),v8),
i(i(v8,v7),v7)),v9)),v9),i(i(i(i(v10,v11),i(v11,v10)),i(v11,v10)),v12)),v12),i(i(v13,i(v14,v13)),v15))),
i(x,v15))),2).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
i(i(v14,i(v15,v14)),v16))),v16)),2).
% % following is an 18-step proof of step 21 of a 26/15 proof of Rezus from the Ulrich 3-basis.
% weight(P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))),2).
% weight(P(i(i(i(x,y),z),i(y,z))),2).
% weight(P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))),2).
% weight(P(i(i(x,y),i(z,i(i(y,u),i(x,u))))),2).
% weight(P(i(x,i(i(i(y,x),i(x,y)),y))),2).
% weight(P(i(i(x,i(i(y,z),i(z,y))),i(z,i(x,y)))),2).
% weight(P(i(i(x,y),i(i(z,x),i(z,y)))),2).
% weight(P(i(x,i(i(x,y),y))),2).
% weight(P(i(i(x,i(i(y,z),u)),i(x,i(z,u)))),2).
% weight(P(i(i(x,i(y,z)),i(y,i(x,z)))),2).
% weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)),2).
% weight(P(i(i(i(x,y),y),i(i(y,x),x))),2).
% weight(P(i(x,i(i(y,i(x,z)),i(y,z)))),2).
% weight(P(i(i(x,i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))),2).
% weight(P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))),2).
% weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)),2).
% weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)),2).
% weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8)),2).
% % Following 8 complete a proof of Rezus from the Ulrich 3-basis, templulrich.rezus.mv.out7g,
% cramming on preceding 18.
% weight(P(i(i(i(x,y),i(y,x)),i(y,x))),2).
% weight(P(i(x,i(i(x,i(i(y,i(z,y)),u)),u))),2).
% weight(P(i(i(x,i(i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6),i(i(i(i(v7,v8),v8),
i(i(v8,v7),v7)),v9)),v9),v10)),i(x,v10))),2).
% weight(P(i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),i(x,u))),2).
% weight(P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)),2).
% weight(P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),
i(i(v12,i(v13,v12)),v14)),v14)),2).
% weight(P(i(i(x,i(i(i(i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6),i(i(i(i(v7,v8),v8),
i(i(v8,v7),v7)),v9)),v9),i(i(i(i(v10,v11),i(v11,v10)),i(v11,v10)),v12)),v12),i(i(v13,i(v14,v13)),v15))),
i(x,v15))),2).
% weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
i(i(v14,i(v15,v14)),v16))),v16)),2).
% % Following 27/15 prove Rezus from the Ulrich 3-basis, temp.ulrich.rezus.mv.out7c.
% weight(P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))),1).
% weight(P(i(i(i(x,y),z),i(y,z))),1).
% weight(P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))),1).
% weight(P(i(i(x,y),i(z,i(i(y,u),i(x,u))))),1).
% weight(P(i(x,i(i(i(y,x),i(x,y)),y))),1).
% weight(P(i(i(x,i(i(y,z),i(z,y))),i(z,i(x,y)))),1).

% weight(P(i(i(x,y),i(i(z,x),i(z,y)))),1).
% weight(P(i(x,i(i(x,y),y))),1).
% weight(P(i(i(x,i(i(y,z),u)),i(x,i(z,u)))),1).
% weight(P(i(i(x,i(y,z)),i(y,i(x,z)))),1).
% weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)),1).
% weight(P(i(i(i(x,y),y),i(i(y,x),x))),1).
% weight(P(i(x,i(i(y,i(x,z)),i(y,z)))),1).
% weight(P(i(i(i(x,y),i(y,x)),i(y,x))),1).
% weight(P(i(i(i(i(i(x,y),y),i(i(y,x),x)),z),z)),1).
% weight(P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))),1).
% weight(P(i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),i(x,u))),1).
% weight(P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))),1).
% weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)),1).
% weight(P(i(x,i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),u))),1).
% weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)),1).
% weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8)),1).
% weight(P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)),1).
% weight(P(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),v12),v12)),1).
% weight(P(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),i(i(v12,i(v13,v12)),v14)),v14)),1).
% weight(P(i(i(x,i(i(i(i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6),i(i(i(i(v7,v8),v8),
i(i(v8,v7),v7)),v9)),v9),i(i(i(i(v10,v11),i(v11,v10)),i(v11,v10)),v12)),v12),i(i(v13,i(v14,v13)),v15))),
i(x,v15))),1).
% weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
i(i(v14,i(v15,v14)),v16))),v16)),1).
% % Following 54 prove Rezus of Ulrich for MV implicational, from the 3-basis,
% % temp.ulrich.rezus.mv.out7a.
% weight(P(i(x,i(y,i(z,y)))),1).
% weight(P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))),1).
% weight(P(i(i(i(x,y),z),i(y,z))),1).
% weight(P(i(i(i(x,i(y,x)),z),i(u,z))),1).
% weight(P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))),1).
% weight(P(i(i(x,y),i(i(i(x,z),u),i(i(y,z),u)))),1).
% weight(P(i(i(x,y),i(z,i(i(y,u),i(x,u))))),1).
% weight(P(i(i(i(x,y),z),i(i(i(u,x),y),z))),1).
% weight(P(i(x,i(i(i(y,x),i(x,y)),y))),1).
% weight(P(i(x,i(i(x,y),i(z,y)))),1).
% weight(P(i(x,i(y,i(z,x)))),1).
% weight(P(i(x,i(i(i(y,z),u),i(i(x,z),u)))),1).
% weight(P(i(i(x,i(y,z)),i(y,i(x,i(u,z))))),1).
% weight(P(i(i(i(i(i(x,y),z),i(i(u,y),z)),v),i(u,v))),1).
% weight(P(i(i(i(x,y),z),i(i(z,u),i(y,u)))),1).
% weight(P(i(x,i(y,i(i(i(z,u),v),i(i(x,u),v))))),1).
% weight(P(i(i(x,i(i(y,z),i(z,y))),i(z,i(x,y)))),1).
% weight(P(i(i(i(x,y),i(y,x)),i(y,i(z,x)))),1).
% weight(P(i(i(i(x,i(y,z)),u),i(i(x,z),u))),1).
% weight(P(i(i(x,i(y,z)),i(y,i(x,z)))),1).
% weight(P(i(i(i(x,y),z),i(u,i(i(u,y),z)))),1).
% weight(P(i(x,i(y,y))),1).

```
% weight(P(i(i(x,y),i(i(z,x),i(z,y)))),1).
% weight(P(i(x,i(i(x,y),y))),1).
% weight(P(i(x,x)),1).
% weight(P(i(i(x,i(i(y,i(z,y)),u)),i(x,i(v,u)))),1).
% weight(P(i(i(x,i(i(y,z),u)),i(x,i(z,u)))),1).
% weight(P(i(i(i(i(x,y),y),z),i(x,z))),1).
% weight(P(i(i(i(x,x),y),y)),1).
% weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)),1).
% weight(P(i(i(i(x,i(y,y)),z),z)),1).
% weight(P(i(i(i(i(x,y),i(i(y,z),i(x,z))),u),u)),1).
% weight(P(i(i(i(x,i(y,x)),z),z)),1).
% weight(P(i(i(x,i(i(y,y),z)),i(x,z))),1).
% weight(P(i(i(x,i(i(y,i(z,z)),u)),i(x,u))),1).
% weight(P(i(i(x,y),i(i(i(z,i(u,z)),x),y))),1).
% weight(P(i(i(i(x,i(y,z)),u),i(i(y,i(x,z)),u))),1).
% weight(P(i(i(i(x,y),i(y,x)),i(y,x))),1).
% weight(P(i(i(i(i(i(x,y),i(y,x)),i(y,x)),z),z)),1).
% weight(P(i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),i(x,u))),1).
% weight(P(i(i(i(i(x,y),i(i(y,z),i(x,z))),i(i(u,i(v,u)),w)),i(v6,w))),1).
% weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),i(v6,w))),1).
% weight(P(i(i(i(x,y),y),i(i(y,x),x))),1).
% weight(P(i(i(i(i(i(x,y),y),i(i(y,x),x)),z),z)),1).
% weight(P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))),1).
% weight(P(i(i(i(i(x,y),i(i(y,z),i(x,z))),i(i(u,i(v,u)),w)),w)),1).
% weight(P(i(x,i(i(x,i(i(y,i(z,y)),u)),u))),1).
% weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)),1).
% weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)),1).
% weight(P(i(x,i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),u))),1).
% weight(P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8)),1).
% weight(P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)),1).
% weight(P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),i(i(v12,i(v13,v12)),v14)),v14)),1).
% weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
  i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16)),1).
end_of_list.

list(usable).
-P(i(x,y)) | -P(x) | P(y).
% -P(i(a,a)) | -P(i(i(a,i(b,c)),i(b,i(a,c)))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(a,i(a,b)),i(a,b))) | $ANS(all).
end_of_list.

list(sos).
% Following Ulrich's 3-basis for MV implicational fragment.
P(i(x,i(y,x))). % simp, MV1
P(i(i(x,y),i(i(y,z),i(x,z)))). % syl, MV2
P(i(i(i(u,v),v),i(i(i(u,v),i(v,u)),u))). % Ulrich's axiom to replace Linearity and Inversion
end_of_list.

list(passive).
% Following 27/15 prove Rezus from the Ulrich 3-basis, temp.ulrich.rezus.mv.out7d.
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% -P(i(i(i(i(a1,a2),i(a3,a2)),a4),i(i(a3,a1),a4))) | $ANS(REZPRF).
% -P(i(i(i(a1,a2),a3),i(a2,a3))) | $ANS(REZPRF).
% -P(i(i(a1,i(a2,a3)),i(i(a4,a2),i(a1,i(a4,a3))))) | $ANS(REZPRF).
% -P(i(i(a1,a2),i(a3,i(i(a2,a4),i(a1,a4))))) | $ANS(REZPRF).
% -P(i(a1,i(i(i(a2,a1),i(a1,a2)),a2))) | $ANS(REZPRF).
% -P(i(i(a1,i(i(a2,a3),i(a3,a2))),i(a3,i(a1,a2)))) | $ANS(REZPRF).
% -P(i(i(a1,a2),i(i(a3,a1),i(a3,a2)))) | $ANS(REZPRF).
% -P(i(a1,i(i(a1,a2),a2))) | $ANS(REZPRF).
% -P(i(i(a1,i(i(a2,a3),a4)),i(a1,i(a3,a4)))) | $ANS(REZPRF).
% -P(i(i(a1,i(a2,a3)),i(a2,i(a1,a3)))) | $ANS(REZPRF).
% -P(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),a4),a4)) | $ANS(REZPRF).
% -P(i(i(i(a1,a2),a2),i(i(a2,a1),a1))) | $ANS(REZPRF).
% -P(i(a1,i(i(a2,i(a1,a3)),i(a2,a3)))) | $ANS(REZPRF).
% -P(i(i(i(a1,a2),i(a2,a1)),i(a2,a1))) | $ANS(REZPRF).
% -P(i(i(i(i(i(a1,a2),a2),i(i(a2,a1),a1)),a3),a3)) | $ANS(REZPRF).
% -P(i(i(a1,i(a2,i(a3,a2)),a4)),i(a1,a4))) | $ANS(REZPRF).
% -P(i(i(a1,i(i(i(i(a2,a3),i(a3,a2)),i(a3,a2)),a4)),i(a1,a4))) | $ANS(REZPRF).
-P(i(i(a1,i(i(i(i(a2,a3),a3),i(i(a3,a2),a2)),a4)),i(a1,a4))) | $ANS(REZPRF).
-P(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6)) | $ANS(REZPRF).
-P(i(a1,i(i(a1,i(i(i(i(a2,a3),i(a3,a2)),i(a3,a2)),a4)),a4))) | $ANS(REZPRF).
-P(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),b6),b6)) | $ANS(REZPRF).
-P(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
  i(i(b7,b6),b6)),b8)),b8)) | $ANS(REZPRF).
-P(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
  i(i(b7,b6),b6)),b8)),b8),i(i(i(b9,b10),i(b10,b9)),i(b10,b9)),b11)),b11)) | $ANS(REZPRF).
-P(i(i(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
  i(i(b7,b6),b6)),b8)),b8),i(i(i(b9,b10),i(b10,b9)),i(b10,b9)),b11)),b11),b12),b12)) | $ANS(REZPRF).
-P(i(i(i(i(i(i(i(i(i(i(a1,a2),i(i(a3,a1),i(a3,a2))),i(i(a4,i(b,a4)),a6)),a6),i(i(i(b6,b7),b7),
  i(i(b7,b6),b6)),b8)),b8),i(i(i(b9,b10),i(b10,b9)),i(b10,b9)),b11)),b11),i(i(b12,i(b13,b12)),b14)),b14)) |
  $ANS(REZPRF).
-P(i(i(a1,i(i(i(i(i(i(i(i(a2,a3),i(i(a4,a2),i(a4,a3))),i(i(b,i(a6,b)),b6)),b6),i(i(i(b7,b8),b8),
  i(i(b8,b7),b7)),b9)),b9),i(i(i(b10,b11),i(b11,b10)),i(b11,b10)),b12)),b12),i(i(b13,i(b14,b13)),b15))),
    i(a1,b15))) | $ANS(REZPRF).
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZPROOF).
% Following is neg of Ulrich's 37-symbol single axiom for the implicational fragment of MV.
-P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)),i(b6,a6)),b7)),
  i(i(b6,b),b7))) | $ANS(UL37PROOF).
% Following is neg of the rezus-style single axiom for the implicational fragment of MV.
-(P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
    i(i(b14,i(b15,b14)),b16))),b16)),2) | $ANS(REZMV).
end_of_list.

list(demodulators).
(P(i(i(x,i(i(y,z),i(z,y))),i(z,i(x,y)))) = junk).
(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)) = junk).
% (P(i(i(i(i(i(x,y),y),i(i(y,x),x)),z),z)) = junk).
% (i(i(x,x),y) = junk).
% (i(y,i(x,x)) = junk).
(i(x,junk) = junk).
(i(junk,x) = junk).
```

(P(junk) = $T).
end_of_list.

A Short Proof Deriving Rezus from the Rezus 3-Basis

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----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on jaguar.mcs.anl.gov,
Wed Jun  4 15:57:01 2008
The command was "otter".  The process ID is 24207.

----> UNIT CONFLICT at   0.08 sec ----> 375 [binary,374.1,14.1] $ANS(REZPROOF).

Length of proof is 23.  Level of proof is 14.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(x,i(y,x))).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(i(u,v),v),i(i(i(u,v),i(v,u)),u))).
14 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZPROOF).
23 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
25 [hyper,1,3,2] P(i(i(i(x,y),z),i(y,z))).
26 [hyper,1,3,4] P(i(i(i(i(i(x,y),i(y,x)),x),z),i(i(i(x,y),y),z))).
28 [hyper,1,23,23] P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))).
34 [hyper,1,25,4] P(i(x,i(i(i(y,x),i(x,y)),y))).
40 [hyper,1,26,25] P(i(i(i(x,y),y),i(i(y,x),x))).
64 [hyper,1,25,40] P(i(x,i(i(x,y),y))).
71 [hyper,1,28,64] P(i(i(x,i(y,z)),i(y,i(x,z)))).
85 [hyper,1,71,71] P(i(x,i(i(y,i(x,z)),i(y,z)))).
95 [hyper,1,71,34] P(i(i(i(x,y),i(y,x)),i(y,x))).
98 [hyper,1,71,3] P(i(i(x,y),i(i(z,x),i(z,y)))).
107 [hyper,1,85,40] P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))).
111 [hyper,1,85,2] P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))).
112 [hyper,1,85,95] P(i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),i(x,u))).
131 [hyper,1,71,107] P(i(x,i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),u))).
137 [hyper,1,71,111] P(i(x,i(i(x,i(i(y,i(z,y)),u)),u))).
204 [hyper,1,137,98] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
262 [hyper,1,131,204] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8)).
282 [hyper,1,85,262] P(i(i(x,i(i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6),
  i(i(i(v7,v8),v8),i(i(v8,v7),v7)),v9)),v9),v10)),i(x,v10))).
284 [hyper,1,282,112] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)).
315 [hyper,1,137,284] P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),
  i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11),
  i(i(v12,i(v13,v12)),v14)),v14)).
338 [hyper,1,98,315] P(i(i(x,i(i(i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6),
  i(i(i(v7,v8),v8),i(i(v8,v7),v7)),v9)),v9),i(i(i(v10,v11),i(v11,v10)),i(v11,v10)),v12)),v12),
  i(i(v13,i(v14,v13)),v15))),i(x,v15))).
374 [hyper,1,111,338] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
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i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
i(i(v14,i(v15,v14)),v16))),v16)).

The reduction in proof length from 54 to 23, with stops along the way with proofs of various lengths, indicates that the approach merits emulation. Of course, you may find a shorter proof. Also, I do not offer here a proof, a short or long one, that derives from the Rezus formula the Ulrich 3-basis.

I then replaced, for my study of the Ulrich 3-basis, the Rezus formula as target with the Ulrich 37-symbol single axiom. In other words, I was attempting to improve upon "A Most Satisfying Proof", that of length 12. I was unable to do so. Perhaps the already-discussed strong connection between the Ulrich 3-basis and his single axiom for *MV*-implicational explains in the main why I could not find a proof of length strictly less than 12 (deduced steps). So I turned to my last basis of study for the implicational fragment of *MV*.

The 3-basis in question is due to Wozniakowska. As noted earlier, I had in hand a 59-step proof, suggesting that perhaps substantial progress could be made in the context of proof shortening. Iteration and the use of demodulation to block thought-to-be unwanted formulas played key roles. Shorter and shorter proofs were found. Rather than presenting the input file that led to the final success, I leave it to you (if curious) to modify or imitate material already offered in this section with the goal of producing an effective input file. By way of a hint, I suggest the following demodulators be used at an appropriate point, and then I give the proof OTTER eventually found.

(P(i(x,i(i(i(y,i(z,y)),u),u))) = junk).
(P(i(i(i(i(i(x,y),y),i(i(y,x),x)),z),z)) = junk).
(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)) = junk).
(P(i(x,i(i(i(y,z),z),i(i(z,y),y)))) = junk).

A Short Proof Deriving the Rezus formula from the Wozniakowska 3-Basis

----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on elephant.mcs.anl.gov,
Sat May 31 20:03:28 2008
The command was "otter". The process ID is 8948.

----> UNIT CONFLICT at 25.27 sec ----> 17278 [binary,17277.1,5.1] $ANS(REZPROOF).

Length of proof is 28. Level of proof is 14.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(u,i(v,u))).
3 [] P(i(i(i(u,v),v),i(i(v,u),u))).
4 [] P(i(i(i(w,x),i(w,y)),i(i(x,w),i(x,y)))).
5 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(i(b11,b12),i(b12,b11)),i(b12,b11)),b13)),b13),
i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZPROOF).
15 [hyper,1,2,2] P(i(x,i(y,i(z,y)))).
18 [hyper,1,2,4] P(i(x,i(i(i(y,z),i(y,u)),i(i(z,y),i(z,u))))).
21 [hyper,1,3,15] P(i(i(i(x,i(y,x)),z),z)).
23 [hyper,1,4,18] P(i(i(x,i(i(y,z),i(y,u))),i(x,i(i(z,y),i(z,u))))).
51 [hyper,1,23,2] P(i(i(x,y),i(i(z,x),i(z,y)))).
107 [hyper,1,51,51] P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))).
117 [hyper,1,51,21] P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))).
119 [hyper,1,51,3] P(i(i(x,i(i(y,z),z)),i(x,i(i(z,y),y)))).

214 [hyper,1,51,117] P(i(i(x,i(y,i(i(z,i(u,z)),v))),i(x,i(y,v)))).
252 [hyper,1,119,2] P(i(x,i(i(x,y),y))).
295 [hyper,1,214,214] P(i(i(x,i(y,i(z,y)),i(i(u,i(v,u)),w))),i(x,w))).
317 [hyper,1,214,23] P(i(i(x,i(i(i(y,z),z),i(i(y,z),u))),i(x,i(z,u)))).
409 [hyper,1,107,252] P(i(x,i(i(y,i(x,z)),i(y,z)))).
918 [hyper,1,409,51] P(i(i(x,i(i(i(y,z),i(i(u,y),i(u,z))),v)),i(x,v))).
932 [hyper,1,409,3] P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))).
1631 [hyper,1,918,409] P(i(i(x,y),i(i(y,z),i(x,z)))).
1641 [hyper,1,918,117] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
1713 [hyper,1,932,51] P(i(i(i(i(x,y),y),z),i(i(i(y,x),x),z))).
2527 [hyper,1,1631,1631] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
2717 [hyper,1,252,1641] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)).
2821 [hyper,1,317,1713] P(i(i(i(i(x,y),y),i(i(y,x),z)),i(x,z))).
3977 [hyper,1,932,2717] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8)).
4166 [hyper,1,2527,2821] P(i(i(i(x,y),i(y,x)),i(y,x))).
6512 [hyper,1,252,3977] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8),v9),v9)).
6795 [hyper,1,409,4166] P(i(i(x,i(i(i(i(y,z),i(z,y)),i(z,y)),u)),i(x,u))).
11120 [hyper,1,6795,6512] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8),i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)).
14630 [hyper,1,409,11120] P(i(i(x,i(i(i(i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6), i(i(i(i(v7,v8),v8),i(i(v8,v7),v7)),v9)),v9),i(i(i(i(v10,v11),i(v11,v10)),i(v11,v10)),v12)),v12),v13)), i(x,v13))).
17277 [hyper,1,295,14630] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7), i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13), i(i(v14,i(v15,v14)),v16))),v16)).

Rather than focusing on what might be called the converse, I completed my studies of *MV*-implicational by seeking a short proof that derives the Ulrich single axiom (37 symbol) from the Wozniakowska 3-basis. Eventually, OTTER returned the following.

A Short Proof Deriving the Ulrich 37-Symbol Single Axiom from the Wozniakowska 3-Basis

----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on octopus.mcs.anl.gov,
Fri Jun 6 18:07:29 2008
The command was "otter". The process ID is 1690.

----> UNIT CONFLICT at 1.80 sec ----> 5174 [binary,5173.1,5.1] $ANS(UL37PROOF).

Length of proof is 34. Level of proof is 18.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(u,i(v,u))).
3 [] P(i(i(i(u,v),v),i(i(v,u),u))).
4 [] P(i(i(i(w,x),i(w,y)),i(i(x,w),i(x,y)))).
5 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(a3,a4),a4),i(i(i(a3,a4),i(a4,a3)),a3)),i(b,a6)), i(b6,a6)),b7)),i(i(b6,b),b7))) | $ANS(UL37PROOF).
14 [hyper,1,2,2] P(i(x,i(y,i(z,y)))).
15 [hyper,1,2,3] P(i(x,i(i(i(y,z),z),i(i(z,y),y)))).

17 [hyper,1,2,4] P(i(x,i(i(i(y,z),i(y,u)),i(i(z,y),i(z,u))))).
20 [hyper,1,3,14] P(i(i(i(x,i(y,x)),z),z)).
22 [hyper,1,4,15] P(i(i(x,i(i(y,z),z)),i(x,i(i(z,y),y)))).
29 [hyper,1,4,17] P(i(i(x,i(i(y,z),i(y,u))),i(x,i(i(z,y),i(z,u))))).
30 [hyper,1,3,17] P(i(i(i(i(i(x,y),i(x,z)),i(i(y,x),i(y,z))),u),u)).
53 [hyper,1,29,2] P(i(i(x,y),i(i(z,x),i(z,y)))).
69 [hyper,1,53,53] P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))).
74 [hyper,1,53,20] P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))).
92 [hyper,1,69,30] P(i(i(i(i(i(x,y),i(x,z)),i(i(y,x),i(y,z))),i(u,v)),i(i(w,u),i(w,v)))).
95 [hyper,1,69,3] P(i(i(i(x,y),y),i(i(z,i(y,x)),i(z,x)))).
120 [hyper,1,74,53] P(i(i(i(x,y),z),i(y,z))).
169 [hyper,1,53,120] P(i(i(x,i(i(y,z),u)),i(x,i(z,u)))).
287 [hyper,1,22,169] P(i(i(i(x,i(y,z)),i(i(u,y),z)),i(i(i(y,z),x),x))).
305 [hyper,1,169,4] P(i(i(i(x,y),i(x,z)),i(x,i(y,z)))).
442 [hyper,1,287,15] P(i(i(i(x,i(i(x,y),y)),z),z)).
478 [hyper,1,305,92] P(i(i(i(i(x,y),i(x,z)),i(i(y,x),i(y,z))),i(i(i(i(y,x),i(y,z)),u),i(i(i(x,y),i(x,z)),u)))).
623 [hyper,1,30,478] P(i(i(i(i(x,y),i(x,z)),u),i(i(i(y,x),i(y,z)),u))).
874 [hyper,1,169,623] P(i(i(i(i(x,y),i(x,z)),u),i(i(y,z),u))).
876 [hyper,1,29,623] P(i(i(i(i(x,y),i(x,z)),i(i(y,x),u)),i(i(i(y,z),i(y,x)),i(i(y,z),u)))).
1014 [hyper,1,874,442] P(i(i(i(i(x,y),y),z),i(x,z))).
1021 [hyper,1,874,74] P(i(i(x,y),i(i(i(z,i(u,z)),x),y))).
1034 [hyper,1,876,287] P(i(i(i(i(x,y),y),i(i(x,y),i(z,x))),i(i(i(x,y),y),i(z,x)))).
1204 [hyper,1,53,1014] P(i(i(x,i(i(i(y,z),z),u)),i(x,i(y,u)))).
1211 [hyper,1,1014,874] P(i(i(x,y),i(i(y,z),i(x,z)))).
1222 [hyper,1,1014,69] P(i(x,i(i(x,i(y,z)),i(i(u,y),i(u,z))))).
1404 [hyper,1,1204,95] P(i(i(i(x,y),y),i(z,i(i(z,i(y,x)),x)))).
1412 [hyper,1,1204,1211] P(i(i(x,i(y,z)),i(y,i(x,z)))).
1597 [hyper,1,1034,1404] P(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x))).
1871 [hyper,1,1222,1597] P(i(i(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x)),i(z,u)),i(i(v,z),i(v,u)))).
2478 [hyper,1,1412,1871] P(i(i(x,y),i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(y,v)),i(x,v)))).
3552 [hyper,1,1211,2478] P(i(i(i(i(i(i(i(x,y),y),i(i(i(x,y),i(y,x)),x)),i(z,u)),i(v,u)),w),i(i(v,z),w))).
5173 [hyper,1,1021,3552] P(i(i(i(x,i(y,x)),i(i(i(i(i(z,u),u),i(i(i(z,u),i(u,z)),z)),i(v,w)),
i(v6,w)),v7)),i(i(v6,v),v7))).

At this point the focus turns to the full Lukasiewicz infinite-valued sentential calculus and the seeking of short proofs. Therefore, the basis of concern is a 4-basis consisting of *MV*1, *MV*2, *MV*3, and *MV*4, the last of the cited four being the only axiom in which negation is present. In that, as noted, the only single axiom in hand is a Rezus-style formula (given to me by Ulrich), this study is much briefer than that focusing on *MV*-implicational. The following proof was found.

A Short Proof Deriving a Rezus-Style Single Zxiom for MV-Full from a 4-Basis

----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on elephant.mcs.anl.gov,
Sun Jun 22 18:01:24 2008
The command was "otter". The process ID is 15640.

----> UNIT CONFLICT at 0.08 sec ----> 255 [binary,254.1,25.1] $ANS(REZMV).

Length of proof is 19. Level of proof is 15.

---------------- PROOF ----------------

```
1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] P(i(x,i(y,x))).
3 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
4 [] P(i(i(i(x,y),y),i(i(y,x),x))).
5 [] P(i(i(n(x),n(y)),i(y,x))).
25 [] -P(i(i(i(a1,i(a2,a1)),i(i(i(i(i(i(i(i(a3,a4),i(i(b,a3),i(b,a4))),i(i(a6,i(b6,a6)),b7)),b7),
  i(i(i(i(b8,b9),b9),i(i(b9,b8),b8)),b10)),b10),i(i(i(n(b11),n(b12)),i(b12,b11)),b13)),b13),
  i(i(b14,i(b15,b14)),b16))),b16)) | $ANS(REZMV).
38 [hyper,1,3,3] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
41 [hyper,1,3,2] P(i(i(i(x,y),z),i(y,z))).
54 [hyper,1,41,4] P(i(x,i(i(x,y),y))).
59 [hyper,1,3,54] P(i(i(i(i(x,y),y),z),i(x,z))).
68 [hyper,1,59,38] P(i(i(x,y),i(i(z,x),i(z,y)))).
78 [hyper,1,59,68] P(i(x,i(i(y,i(x,z)),i(y,z)))).
80 [hyper,1,54,68] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),u),u)).
89 [hyper,1,78,5] P(i(i(x,i(i(i(n(y),n(z)),i(z,y)),u)),i(x,u))).
91 [hyper,1,78,4] P(i(i(x,i(i(i(i(y,z),z),i(i(z,y),y)),u)),i(x,u))).
93 [hyper,1,78,2] P(i(i(x,i(i(y,i(z,y)),u)),i(x,u))).
127 [hyper,1,93,80] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
140 [hyper,1,54,127] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),v6),v6)).
154 [hyper,1,91,140] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8)).
165 [hyper,1,54,154] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),v9),v9)).
178 [hyper,1,89,165] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),i(i(i(n(v9),n(v10)),i(v10,v9)),v11)),v11)).
201 [hyper,1,54,178] P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),i(i(i(n(v9),n(v10)),i(v10,v9)),v11)),v11),v12),v12)).
214 [hyper,1,93,201] P(i(i(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7),
  i(i(v7,v6),v6)),v8)),v8),i(i(i(n(v9),n(v10)),i(v10,v9)),v11)),v11),i(i(v12,i(v13,v12)),v14)),v14)).
242 [hyper,1,68,214] P(i(i(x,i(i(i(i(i(i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6)),v6),
  i(i(i(i(v7,v8),v8),i(i(v8,v7),v7)),v9)),v9),i(i(i(n(v10),n(v11)),i(v11,v10)),v12)),v12),
  i(i(v13,i(v14,v13)),v15))),i(x,v15))).
254 [hyper,1,93,242] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
  i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16)).
```

This proof offers to some an interesting property, namely, no formula in the proof relies on double negation. That occurrence was by no means accidental. Indeed, the input file included a demodulator that blocked the retention of any deduced conclusion that contained a term of the form *n*(*n*(*t*)) for some term *t*. I often impose this constraint because numerous experiments suggest that a proof can be completed in far less CPU time. I leave it to you to determine whether a proof shorter than length 19 exists, with or without this restriction on double negation. I also leave to you the production of a appropriate input file. For a hint in that regard, I did rely on resonators that correspond to a 35-step proof (given earlier in this notebook).

As for the converse, that of finding a short proof deriving the 4-basis from the Rezus-style axiom deduced as the last step in the just-given proof, I made little progress. Still, I now give the minor evidence of progress, a 17-step proof that you can compare with the earlier-given 18-step proof.

A Shorter Proof that Derives from the Rezus Axiom for MV-Full a 4-Basis

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on octopus.mcs.anl.gov,
Thu Jun 19 04:10:16 2008
```

The command was "otter". The process ID is 9015.

-----> EMPTY CLAUSE at 30.02 sec ----> 32210 [hyper,2,14,30710,32,26] $ANS(MVALL).

Length of proof is 17. Level of proof is 11.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] -P(i(a,i(b,a))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(i(a,b),b),i(i(b,a),a))) | -P(i(i(n(a),n(b)),i(b,a))) | $ANS(MVALL).
3 [] P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9), i(i(v9,v8),v8)),v10)),v10),i(i(i(n(v11),n(v12)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),v16)).
13 [hyper,1,3,3] P(i(x,i(y,i(z,y)))).
14 [hyper,1,13,13] P(i(x,i(y,x))).
16 [hyper,1,3,13] P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8),i(i(i(n(v9),n(v10)),i(v10,v9)),v11)),v11)).
20 [hyper,1,16,14] P(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(i(v6,v7),v7), i(i(v7,v6),v6)),v8)),v8)).
21 [hyper,1,16,13] P(i(x,i(i(n(y),n(z)),i(z,y)))).
23 [hyper,1,20,14] P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)).
24 [hyper,1,20,13] P(i(x,i(i(i(y,z),z),i(i(z,y),y)))).
26 [hyper,1,21,21] P(i(i(n(x),n(y)),i(y,x))).
31 [hyper,1,23,14] P(i(i(x,y),i(i(z,x),i(z,y)))).
32 [hyper,1,24,24] P(i(i(i(x,y),y),i(i(y,x),x))).
36 [hyper,1,31,31] P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))).
48 [hyper,1,32,24] P(i(i(i(i(i(x,y),y),i(i(y,x),x)),z),z)).
85 [hyper,1,23,48] P(i(x,i(i(x,y),y))).
113 [hyper,1,36,85] P(i(x,i(i(y,i(x,z)),i(y,z)))).
152 [hyper,1,113,113] P(i(i(x,i(i(y,i(i(z,i(y,u)),i(z,u))),v)),i(x,v))).
22413 [hyper,1,152,113] P(i(i(x,i(y,z)),i(y,i(x,z)))).
30710 [hyper,1,22413,31] P(i(i(x,y),i(i(y,z),i(x,z)))).

7. Notes, Comments, and Unplanned Tests of the Subformula Strategy

In this section I present various observations, comments, and notes. Some of what you find in this section was to me a surprise, experiments not planned for this notebook. However, because of questions and (in effect) challenges from my colleague Beeson, I made certain experiments that merit discussion and that lead to an anecdote of possible interest.

In an e-mail, Beeson, being familiar with my use of resonators taken from papers some written many decades ago, wondered what I would have done under the following conditions. Lukasiewicz in the 1930s provided important elements to various areas of logic. Among them were what he called theses (theorems), numbered 4 through 71; the first three (which I give shortly) are an axiom system for classical propositional calculus. As history notes, virtually all of his written work was lost, in part because of the war. However, a key manuscript was saved. Beeson wondered what I would have done if I had never heard of—or they had been lost—theses 4-71. Indeed, as you find in some of my writings, I have frequently used theses 4-71, each included in a weight template with a small assigned value, to direct OTTER's reasoning. Phrased more technically, those sixty-eight formulas have been used as resonators in, for example, my obtaining a proof of the following axiom system for classical propositional calculus.

% Following are the members of the so-called Church system.
P(i(x,i(y,x))).
P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
P(i(i(n(x),n(y)),i(y,x))).

And here I correct an error of mine made many years ago: namely, the given axiom system is due to Lukasiewicz, and not to Church. The system, as Ulrich informs me, was offered by Lukasiewicz as a simplification system of another axiom system due to Frege.

The following (more familiar) axiom system, again due to Lukasiewicz, was used to deduce the three formulas just given, which are, respectively, theses 18, 35, and 49.

```
% Luka 1 2 3.
P(i(i(x,y),i(i(y,z),i(x,z)))).
P(i(i(n(x),x),x)).
P(i(x,i(n(x),y))).
```

A glance at the three theses correctly suggests that thesis 35 is the hardest to prove. In fact, at various times I have offered the corresponding problem as a test of new ideas in the context of automated reasoning. Summarizing, Beeson, out of at least curiosity, asked how, without the sixty-eight theses, I might have proceeded in my attempt to deduce what I had called the Church three-axiom system from the just-given Lukasiewicz three-axiom system.

My thought was to try to obtain the desired proof by relying on some subformulas of thesis 35 and perhaps other subformulas. I by hand selected the following two subformulas of thesis 35 and decided to also rely on the three theses 18, 35, and 49 as resonators. I then submitted the following input file for OTTER's consideration.

An Input File to Prove from a Lukasiewicz Three-Axiom System from Another Lukasiewicz Three-Axiom System

```
% Trying for a shorter proof than 24 of Hilbert from Luka 1 2 3.
set(hyper_res).
assign(max_weight,28).
assign(max_proofs,-1).
clear(print_kept).
% set(process_input).
% set(ancestor_subsume).
set(back_sub).
clear(print_back_sub).
assign(max_distinct_vars,5).
assign(pick_given_ratio,3).
assign(max_mem,800000).
% assign(max_seconds,1).
assign(max_given,1000).
assign(report,3600).
set(order_history).
set(input_sos_first).
set(keep_hint_subsumers).
assign(bsub_hint_wt,1).
assign(heat,0).
assign(dynamic_heat_weight,0).

weight_list(pick_and_purge).
% Following 2 are subformulas of thesis 35.
weight(i(x,i(y,z)),2).
weight(i(i(x,y),i(x,z)),2).
% Following 3 are 18 35 49, the so-called Church system.
weight(P(i(y,i(x,y))),3).
weight(P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))),3).
```

```
weight(P(i(i(n(x),n(y)),i(y,x))),3).
end_of_list.

list(usable).
%  condensed detachment
-P(i(x,y)) | -P(x) | P(y).
%  The following disjunctions are known axiom systems.
-P(i(q,i(p,q))) | -P(i(i(p,i(q,r)),i(i(p,q),i(p,r)))) | -P(i(n(n(p)),p)) | -P(i(p,n(n(p)))) |
  -P(i(i(p,q),i(n(q),n(p)))) | -P(i(i(p,i(q,r)),i(q,i(p,r)))) | $ANS(step_allFrege_18_35_39_40_46_21).
% 21 is dependent.
-P(i(q,i(p,q))) | -P(i(i(p,i(q,r)),i(q,i(p,r)))) | -P(i(i(q,r),i(i(p,q),i(p,r)))) | -P(i(p,i(n(p),q))) |
  -P(i(i(p,q),i(i(n(p),q),q))) | -P(i(i(p,i(p,q)),i(p,q))) | $ANS(step_allHilbert_18_21_22_3_54_30).
% 30 is dependent.
-P(i(q,i(p,q))) | -P(i(i(p,i(q,r)),i(i(p,q),i(p,r)))) | -P(i(i(n(p),n(q)),i(q,p))) |
  $ANS(step_allBEH_Church_FL_18_35_49).
%  -P(i(i(i(p,q),r),i(q,r))) | -P(i(i(i(p,q),r),i(n(p),r))) | -P(i(i(n(p),r),i(i(q,r),i(i(p,q),r)))) |
  $ANS(step_allLuka_x_19_37_59).
-P(i(i(i(p,q),r),i(q,r))) | -P(i(i(i(p,q),r),i(n(p),r))) | -P(i(i(s,i(n(p),r)),i(s,i(i(q,r),i(i(p,q),r))))) |
  $ANS(step_allWos_x_19_37_60).
end_of_list.

list(sos).
%  Luka 1 2 3.
P(i(i(x,y),i(i(y,z),i(x,z)))).
P(i(i(n(x),x),x)).
P(i(x,i(n(x),y))).
end_of_list.

list(passive).
%  Following 3 are negs of 18 35 49, the so-called Church system.
-P(i(q,i(p,q))) | $ANS(th_18).
-P(i(i(p,i(q,r)),i(i(p,q),i(p,r)))) | $ANS(th_35).
-P(i(i(n(p),n(q)),i(q,p))) | $ANS(th_49).
end_of_list.

list(demodulators).
(n(n(n(x))) = junk).
%  (n(n(x)) = junk).
(i(i(x,x),y) = junk).
(i(y,i(x,x)) = junk).
(i(n(i(x,x)),y) = junk).
(i(y,n(i(x,x))) = junk).
(i(x,junk) = junk).
(i(junk,x) = junk).
(n(junk) = junk).
(P(junk) = $T).
end_of_list.

%  list(hints).
%  P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
%  P(i(i(i(n(x),y),z),i(x,z))).
%  P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))).
%  P(i(i(x,n(y)),i(y,i(x,z)))).
```

```
% P(i(x,x)).
% P(i(i(x,y),i(i(n(i(y,z)),i(y,z)),i(x,z)))).
% P(i(i(x,i(n(i(y,z)),i(y,z))),i(i(u,y),i(x,i(u,z))))).
% P(i(x,i(y,i(n(x),z)))).
% P(i(x,i(n(i(i(n(y),y),y)),z))).
% P(i(i(x,i(n(y),y)),i(z,i(x,y)))).
% P(i(i(n(x),y),i(z,i(i(y,x),x)))).
% P(i(i(x,i(y,z)),i(i(n(z),y),i(x,z)))).
% P(i(i(x,i(n(y),z)),i(i(u,i(z,y)),i(x,i(u,y))))).
% P(i(i(n(x),n(y)),i(y,x))).
% P(i(x,i(y,x))).
% P(i(i(x,i(y,z)),i(y,i(x,z)))).
% P(i(n(x),i(x,y))).
% P(i(i(x,y),i(i(z,x),i(z,y)))).
% P(i(i(n(x),y),i(n(y),x))).
% P(i(i(x,i(y,z)),i(i(n(y),z),i(x,z)))).
% P(i(i(n(x),y),i(i(x,y),y))).
% P(i(i(x,y),i(i(n(x),y),y))).
% P(i(i(x,i(x,y)),i(x,y))).
% end_of_list.

list(hot).
-P(i(x,y)) | -P(x) | P(y).
% Luka 1 2 3.
P(i(i(x,y),i(i(y,z),i(x,z)))).
P(i(i(n(x),x),x)).
P(i(x,i(n(x),y))).
end_of_list.
```

Before I present the results of using this input file, I pause for some comments, to permit you, if you wish, to experiment with the file and experience what occurred. In effect quoting from a recent e-mail I received from Ulrich, I suggest that if you do conduct the experiment, you stay close to your computer. In the input file, among other aspects two might be interesting. First, I placed a limit on the number of "given" clauses with the inclusion of assign(max_given,1000). In doing so, I instructed the program to cease reasoning immediately after 1000 clauses were chosen to initiate inference-rule application, in this case, condensed detachment. You can thus limit what occurs in an experiment by means other than placing an upper bound on the number of CPU-seconds used or an upper bound on the amount of memory used (measured in kilobytes).

The second item concerns the use of demodulation to prevent the program from retaining a type of conclusion that might, if kept, cost in CPU time, or worse. The inclusion of the demodulator, (n(n(n(x))) = junk), instructs the program to discard upon generation any new item that contains a subexpression (not necessarily proper) that relies on triple negation. You no doubt recall that, typically, I block retention of so-called double-negation deduced conclusions. In this case, however, that choice might, just perhaps, cloud the analysis of the experiment.

Well, the time has come for the presentation of the results of the attempt (with the last given input file) to prove from Lukasiewicz 1, 2, and 3 the join of theses 18, 35, and 49. As you will immediately see, the corresponding theorem, rather than proving difficult to prove, was easy to prove.

An Unexpectedly Easy-to-Complete Proof

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on elephant.mcs.anl.gov,
```

Sun Jul 27 15:22:19 2008
The command was "otter". The process ID is 32359.

-----> EMPTY CLAUSE at 1.50 sec ----> 7196 [hyper,4,699,7049,5093]
$ANS(step_allBEH_Church_FL_18_35_49).

Length of proof is 42. Level of proof is 23.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
4 [] -P(i(q,i(p,q))) | -P(i(i(p,i(q,r)),i(i(p,q),i(p,r)))) | -P(i(i(n(p),n(q)),i(q,p))) |
$ANS(step_allBEH_Church_FL_18_35_49).
6 [] P(i(i(x,y),i(i(y,z),i(x,z)))).
7 [] P(i(i(n(x),x),x)).
8 [] P(i(x,i(n(x),y))).
25 [hyper,1,6,6] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
26 [hyper,1,6,7] P(i(i(x,y),i(i(n(x),x),y))).
28 [hyper,1,6,8] P(i(i(i(n(x),y),z),i(x,z))).
33 [hyper,1,25,25] P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))).
45 [hyper,1,6,33] P(i(i(i(i(x,y),i(z,i(x,u))),v),i(i(z,i(y,u)),v))).
49 [hyper,1,33,8] P(i(i(x,n(y)),i(y,i(x,z)))).
51 [hyper,1,33,26] P(i(i(x,i(n(y),y)),i(i(y,z),i(x,z)))).
54 [hyper,1,6,26] P(i(i(i(i(n(x),x),y),z),i(i(x,y),z))).
60 [hyper,1,26,8] P(i(i(n(x),x),i(n(x),y))).
63 [hyper,1,33,49] P(i(i(x,y),i(i(z,n(y)),i(x,i(z,u))))).
66 [hyper,1,6,49] P(i(i(i(x,i(y,z)),u),i(i(y,n(x)),u))).
91 [hyper,1,28,7] P(i(x,x)).
268 [hyper,1,66,51] P(i(i(n(x),n(y)),i(i(x,z),i(y,z)))).
292 [hyper,1,28,268] P(i(x,i(i(x,y),i(z,y)))).
313 [hyper,1,33,292] P(i(i(x,i(y,z)),i(y,i(x,i(u,z))))).
338 [hyper,1,313,292] P(i(i(x,y),i(x,i(z,i(u,y))))).
354 [hyper,1,313,8] P(i(n(x),i(x,i(y,z)))).
370 [hyper,1,6,354] P(i(i(i(x,i(y,z)),u),i(n(x),u))).
517 [hyper,1,338,91] P(i(x,i(y,i(z,x)))).
539 [hyper,1,63,517] P(i(i(x,n(i(y,i(z,u)))),i(u,i(x,v)))).
540 [hyper,1,51,517] P(i(i(i(x,y),z),i(y,z))).
548 [hyper,1,6,517] P(i(i(i(x,i(y,z)),u),i(z,u))).
675 [hyper,1,539,91] P(i(x,i(n(i(y,i(z,x))),u))).
677 [hyper,1,313,675] P(i(n(i(x,i(y,z))),i(z,i(u,v)))).
699 [hyper,1,7,677] P(i(x,i(y,x))).
754 [hyper,1,699,7] P(i(x,i(i(n(y),y),y))).
835 [hyper,1,548,60] P(i(x,i(n(i(y,x)),z))).
969 [hyper,1,33,754] P(i(i(x,i(n(y),y)),i(z,i(x,y)))).
1191 [hyper,1,370,51] P(i(n(x),i(i(y,z),i(x,z)))).
1218 [hyper,1,33,1191] P(i(i(x,i(y,z)),i(n(u),i(x,i(u,z))))).
1333 [hyper,1,1218,835] P(i(n(x),i(y,i(x,z)))).
1356 [hyper,1,6,1333] P(i(i(i(x,i(y,z)),u),i(n(y),u))).
1552 [hyper,1,66,969] P(i(i(n(x),n(y)),i(z,i(y,x)))).
4601 [hyper,1,969,1356] P(i(x,i(i(i(y,i(z,u)),z),z))).
4836 [hyper,1,4601,4601] P(i(i(i(x,i(y,z)),y),y)).
4886 [hyper,1,54,4836] P(i(i(i(x,y),x),x)).
4971 [hyper,1,25,4886] P(i(i(x,i(x,y)),i(x,y))).

5082 [hyper,1,45,4971] P(i(i(i(x,y),i(y,z)),i(i(x,y),i(x,z)))).
5093 [hyper,1,4971,1552] P(i(i(n(x),n(y)),i(y,x))).
5165 [hyper,1,540,5082] P(i(i(x,y),i(i(z,x),i(z,y)))).
5336 [hyper,1,5165,4971] P(i(i(x,i(y,i(y,z))),i(x,i(y,z)))).
7049 [hyper,1,45,5336] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).

One aspect that was predicted was present: the subproof of thesis 35 was completed after the other two theses were proved. In other words, of the three theses to prove, thesis 35 was much harder to prove than were either of the other two. Indeed, the first two were proven in less than 1 CPU-seconds, and the third in just under 1.5 CPU-seconds; the respective lengths of the proofs are 18, 36, and 40, proving in order thesis 18, 49, and 35. I claim, not profoundly, that more evidence was just given for the power of the subformula strategy, and, yes, I was unprepared for the speed with which the goal was reached.

Naturally, you would like to know what occurred when the two subformulas were replaced by theses 4 through 71. Approximately five times as much CPU time was required (on the same computer) to reach the goal. I am almost certain you will not consider the comparison to be of little significance because of the small amount of computer time involved. By the way, the resulting proof of the join of the three targets has length 34. To verify what has just been offered, you need merely replace in the input file the two subformulas by resonators corresponding to theses 4-71, the following.

```
% Following are theses 4 through 71.
weight(P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))),2).
weight(P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))),2).
weight(P(i(i(x,y),i(i(i(x,z),u),i(i(y,z),u)))),2).
weight(P(i(i(x,i(i(y,z),u)),i(i(y,v),i(x,i(i(v,z),u))))),2).
weight(P(i(i(x,y),i(i(z,x),i(i(y,u),i(z,u))))),2).
weight(P(i(i(i(n(x),y),z),i(x,z))),2).
weight(P(i(x,i(i(i(n(x),x),x),i(i(y,x),x)))),2).
weight(P(i(i(x,i(i(n(y),y),y)),i(i(n(y),y),y))),2).
weight(P(i(x,i(i(n(y),y),y))),2).
weight(P(i(i(n(x),y),i(z,i(i(y,x),x)))),2).
weight(P(i(i(i(x,i(i(y,z),z)),u),i(i(n(z),y),u))),2).
weight(P(i(i(n(x),y),i(i(y,x),x))),2).
weight(P(i(x,x)),2).
weight(P(i(x,i(i(y,x),x))),2).
weight(P(i(x,i(y,x))),2).
weight(P(i(i(i(x,y),z),i(y,z))),2).
weight(P(i(x,i(i(x,y),y))),2).
weight(P(i(i(x,i(y,z)),i(y,i(x,z)))),2).
weight(P(i(i(x,y),i(i(z,x),i(z,y)))),2).
weight(P(i(i(i(x,i(y,z)),u),i(i(y,i(x,z)),u))),2).
weight(P(i(i(i(x,y),x),x)),2).
weight(P(i(i(i(x,y),z),i(i(x,u),i(i(u,y),z)))),2).
weight(P(i(i(i(x,y),z),i(i(z,x),x))),2).
weight(P(i(i(i(x,y),y),i(i(y,x),x))),2).
weight(P(i(i(i(i(x,y),y),z),i(i(i(y,u),x),z))),2).
weight(P(i(i(i(x,y),z),i(i(x,z),z))),2).
weight(P(i(i(x,i(x,y)),i(x,y))),2).
weight(P(i(i(x,y),i(i(i(x,z),u),i(i(y,u),u)))),2).
weight(P(i(i(i(x,y),z),i(i(x,u),i(i(u,z),z)))),2).
weight(P(i(i(x,y),i(i(y,i(z,i(x,u))),i(z,i(x,u))))),2).
weight(P(i(i(x,i(y,i(z,u))),i(i(z,x),i(y,i(z,u))))),2).
weight(P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))),2).
weight(P(i(n(x),i(x,y))),2).
weight(P(i(i(i(x,y),z),i(n(x),z))),2).
```

```
weight(P(i(i(x,n(x)),n(x))),2).
weight(P(i(n(n(x)),x)),2).
weight(P(i(x,n(n(x)))),2).
weight(P(i(i(x,y),i(n(n(x)),y))),2).
weight(P(i(i(i(n(n(x)),y),z),i(i(x,y),z))),2).
weight(P(i(i(x,y),i(i(y,n(x)),n(x)))),2).
weight(P(i(i(x,i(y,n(z))),i(i(z,y),i(x,n(z))))),2).
weight(P(i(i(x,i(y,z)),i(i(n(z),y),i(x,z)))),2).
weight(P(i(i(x,y),i(n(y),n(x)))),2).
weight(P(i(i(x,n(y)),i(y,n(x)))),2).
weight(P(i(i(n(x),y),i(n(y),x))),2).
weight(P(i(i(n(x),n(y)),i(y,x))),2).
weight(P(i(i(i(n(x),y),z),i(i(n(y),x),z))),2).
weight(P(i(i(x,i(y,z)),i(x,i(n(z),n(y))))),2).
weight(P(i(i(x,i(y,n(z))),i(x,i(z,n(y))))),2).
weight(P(i(i(n(x),y),i(i(x,y),y))),2).
weight(P(i(i(x,y),i(i(n(x),y),y))),2).
weight(P(i(i(x,y),i(i(x,n(y)),n(x)))),2).
weight(P(i(i(i(i(x,y),y),z),i(i(n(x),y),z))),2).
weight(P(i(i(n(x),y),i(i(x,z),i(i(z,y),y)))),2).
weight(P(i(i(i(i(x,y),i(i(y,z),z)),u),i(i(n(x),z),u))),2).
weight(P(i(i(n(x),y),i(i(z,y),i(i(x,z),y)))),2).
weight(P(i(i(x,i(n(y),z)),i(x,i(i(u,z),i(i(y,u),z))))),2).
weight(P(i(i(x,y),i(i(z,y),i(i(n(x),z),y)))),2).
weight(P(i(i(n(n(x)),y),i(x,y))),2).
weight(P(i(x,i(y,y))),2).
weight(P(i(n(i(x,x)),y)),2).
weight(P(i(i(n(x),n(i(y,y))),x)),2).
weight(P(i(n(i(x,y)),x)),2).
weight(P(i(n(i(x,y)),n(y))),2).
weight(P(i(n(i(x,n(y))),y)),2).
weight(P(i(x,i(n(y),n(i(x,y))))),2).
weight(P(i(x,i(y,n(i(x,n(y)))))),2).
weight(P(n(i(i(x,x),n(i(y,y))))),2).
```

If you do not include the two subformulas and do not replace them with anything, such as the sixty-eight resonators, you will experience the difficulty of proving thesis 35. Indeed, when I conducted that experiment, it experiment in focus ran out of memory without proving thesis 35. Resumption, for another anecdote in the context of the subformula strategy, of this section is in order.

My second anecdote in the context of the subformula strategy concerns both Beeson and Ulrich and a different area of logic, namely, the implicational fragment of intuitionistic logic (or intuitionistic propositional calculus), sometimes called *positive implication*. In particular, Beeson sent me two single axioms, from the research of Meredith, for this area and suggested I seek a short proof for each with as hypothesis a 4-basis from Hilbert. All will be shown shortly. Ulrich sent me from his own research ten additional single axioms for positive implication. One of my experiments, relying on the following input file, sought to prove all twelve single axioms from the Hilbert 4-basis in one run and, at the same time, provide yet another test of the subformula strategy.

An Input File for the Study of Positive Implication

```
% Otter file, by Beeson
% Starting with the Hilbert 4-base (1922) for
% positive implicational calculus, derive other bases for that logic.
```

```
set(hyper_res).
assign(max_weight,6).
clear(print_kept).
% set(process_input).
% set(ancestor_subsume).
set(back_sub).
clear(print_back_sub).
assign(max_distinct_vars,6).
% assign(pick_given_ratio,2).
assign(max_mem,800000).
% assign(max_seconds,3).
% assign(max_given,1200).
assign(max_proofs,-1).
assign(report,3600).
set(order_history).
set(input_sos_first).
set(sos_queue).
set(keep_hint_subsumers).
assign(bsub_hint_wt,1).

weight_list(pick_and_purge).
% Following 68 sorted subformulas of ten Ulrich single axioms for positive implicational.
weight(P(i(i(i(u,v),w),i(i(v,i(w,x)),i(y,i(v,x))))),1).
weight(P(i(i(i(u,v),w),i(x,i(i(w,i(w,y)),i(v,y))))),1).
weight(P(i(i(u,i(v,w)),i(i(i(x,u),v),i(y,i(u,w))))),1).
weight(P(i(i(u,i(v,w)),i(x,i(i(i(y,u),v),i(u,w))))),1).
weight(P(i(i(u,v),i(i(i(w,i(x,u)),i(v,y)),i(u,y)))),1).
weight(P(i(i(u,v),i(i(v,i(i(w,u),x)),i(y,i(u,x))))),1).
weight(P(i(i(u,v),i(w,i(i(i(x,u),i(v,y)),i(u,y))))),1).
weight(P(i(u,i(i(i(v,w),x),i(i(w,i(x,y)),i(w,y))))),1).
weight(P(i(u,i(i(i(v,w),x),i(i(x,i(x,y)),i(w,y))))),1).
weight(P(i(u,i(i(v,i(w,x)),i(i(i(y,v),w),i(v,x))))),1).
weight(i(i(i(u,v),w),i(i(v,i(w,x)),i(y,i(v,x)))),1).
weight(i(i(i(u,v),w),i(x,i(i(w,i(w,y)),i(v,y)))),1).
weight(i(i(i(v,w),x),i(i(w,i(x,y)),i(w,y))),1).
weight(i(i(i(v,w),x),i(i(x,i(x,y)),i(w,y))),1).
weight(i(i(i(w,i(x,u)),i(v,y)),i(u,y)),1).
weight(i(i(i(x,u),i(v,y)),i(u,y)),1).
weight(i(i(i(x,u),v),i(y,i(u,w))),1).
weight(i(i(i(y,u),v),i(u,w)),1).
weight(i(i(i(y,v),w),i(v,x)),1).
weight(i(i(u,i(v,w)),i(i(i(x,u),v),i(y,i(u,w)))),1).
weight(i(i(u,i(v,w)),i(x,i(i(i(y,u),v),i(u,w)))),1).
weight(i(i(u,v),i(i(i(w,i(x,u)),i(v,y)),i(u,y))),1).
weight(i(i(u,v),i(i(v,i(i(w,u),x)),i(y,i(u,x)))),1).
weight(i(i(u,v),i(w,i(i(i(x,u),i(v,y)),i(u,y)))),1).
weight(i(i(u,v),w),1).
weight(i(i(v,i(i(w,u),x)),i(y,i(u,x))),1).
weight(i(i(v,i(w,x)),i(i(i(y,v),w),i(v,x))),1).
weight(i(i(v,i(w,x)),i(y,i(v,x))),1).
weight(i(i(v,w),x),1).
weight(i(i(w,i(w,y)),i(v,y)),1).
weight(i(i(w,i(x,u)),i(v,y)),1).
```

```
weight(i(i(w,i(x,y)),i(w,y)),1).
weight(i(i(w,u),x),1).
weight(i(i(x,i(x,y)),i(w,y)),1).
weight(i(i(x,u),i(v,y)),1).
weight(i(i(x,u),v),1).
weight(i(i(y,u),v),1).
weight(i(i(y,v),w),1).
weight(i(u,i(i(i(v,w),x),i(i(w,i(x,y)),i(w,y)))),1).
weight(i(u,i(i(i(v,w),x),i(i(x,i(x,y)),i(w,y)))),1).
weight(i(u,i(i(v,i(w,x)),i(i(i(y,v),w),i(v,x)))),1).
weight(i(u,i(v,w)),1).
weight(i(u,v),1).
weight(i(u,w),1).
weight(i(u,x),1).
weight(i(u,y),1).
weight(i(v,i(i(w,u),x)),1).
weight(i(v,i(w,x)),1).
weight(i(v,w),1).
weight(i(v,x),1).
weight(i(v,y),1).
weight(i(w,i(i(i(x,u),i(v,y)),i(u,y))),1).
weight(i(w,i(w,y)),1).
weight(i(w,i(x,u)),1).
weight(i(w,i(x,y)),1).
weight(i(w,u),1).
weight(i(w,x),1).
weight(i(w,y),1).
weight(i(x,i(i(i(y,u),v),i(u,w))),1).
weight(i(x,i(i(w,i(w,y)),i(v,y))),1).
weight(i(x,i(x,y)),1).
weight(i(x,u),1).
weight(i(x,y),1).
weight(i(y,i(u,w)),1).
weight(i(y,i(u,x)),1).
weight(i(y,i(v,x)),1).
weight(i(y,u),1).
weight(i(y,v),1).
end_of_list.

list(usable).
% condensed detachment
-P(i(x,y)) | -P(x) | P(y).
end_of_list.

list(sos).
%  Hilbert's 4-base for positive implicational logic  (1922)
P(i(i(x,i(x,y)),i(x,y))).       %H1
P(i(i(y,z),i(i(x,y),i(x,z)))).  %H2
P(i(i(x,i(y,z)),i(y,i(x,z)))).  %H3
P(i(x,i(y,x))).                 %H4
end_of_list.

list(passive).
```

```
% Following 10 negations of Ulrich single axioms.
-P(i(i(a4,i(a5,a6)),i(a1,i(i(i(a2,a4),a5),i(a4,a6))))) | $ANS(UL01). % HI-3
-P(i(a4,i(i(a5,i(a6,a1)),i(i(i(a2,a5),a6),i(a5,a1))))) | $ANS(UL02). % HI-4
-P(i(i(i(a4,a5),a6),i(i(a5,i(a6,a1)),i(a2,i(a5,a1))))) | $ANS(UL03). % HI-5
-P(i(i(a4,i(a5,a6)),i(i(i(a1,a4),a5),i(a2,i(a4,a6))))) | $ANS(UL04). % HI-6
-P(i(a4,i(i(i(a5,a6),a1),i(i(a6,i(a1,a2)),i(a6,a2))))) | $ANS(UL05). % HI-7
-P(i(i(a4,a5),i(i(i(a6,i(a1,a4)),i(a5,a2)),i(a4,a2)))) | $ANS(UL06). % HI-8
-P(i(i(a4,a5),i(i(a5,i(i(a6,a4),a1)),i(a2,i(a4,a1))))) | $ANS(UL07). % HI-9
-P(i(i(a4,a5),i(a6,i(i(i(a1,a4),i(a5,a2)),i(a4,a2))))) | $ANS(UL08). % HI-10
-P(i(a4,i(i(i(a5,a6),a1),i(i(a1,i(a1,a2)),i(a6,a2))))) | $ANS(UL09). % HI-11
-P(i(i(i(a4,a5),a6),i(a1,i(i(a6,i(a6,a2)),i(a5,a2))))) | $ANS(UL10). % HI-12
% Following are Meredith's two single axioms
-P(i(i(i(p,q),r),i(s,i(i(q,i(r,t)),i(q,t))))) | $ANS(Meredith_1A).
-P(i(t,i(i(p,q),i(i(i(s,p),i(q,r)),i(p,r))))) | $ANS(Meredith_1B).
end_of_list.
```

This test of the strategy is different from many I have discussed in various notebooks in that sixty-eight subformulas are in use, far more (if I recall correctly) than previously in focus. Therefore, I was indeed aware that such an abundance might be detrimental to OTTER's performance. However, all went extremely well, as the following data shows. With the retention of clause (78518) and the expenditure of just under 762 CPU-seconds, all twelve targets were reached: all twelve single axioms were deduced from the Hilbert 4-basis found in list(sos) in the preceding input file. The shortest proof has length 10, and the longest length 19; the level of the proofs varies from 4 to 6. If you take all of the deduced steps, 164 of them, and sort to remove duplicates, you find that 69 formulas suffice to prove all twelve single axioms as theorems. Of the proofs, Ulrich's ninth single axiom (of his ten) was proved in ten applications of condensed detachment; his third was proved with nineteen applications. Rather than giving the twelve proofs, I leave to you the finding of them, with the given input file, or with some other approach with some other program or by hand. Also, I have not as yet gone the other direction, that is, proved the Hilbert 4-basis from each of the twelve single axioms.

Of course, given the various successes resulting from use of the subformula strategy, almost assuredly you would like an example of its use resulting in failure. The area of concern is equivalential calculus, an area that admits various axiomatizations including the following.

```
P(e(e(x,y),e(y,x))). % symmetry
P(e(e(x,y),e(e(y,z),e(x,z)))). % transitivity
```

You might expect, since equivalence is in focus, that a third formula would be needed, the following for reflexivity.

```
P(e(x,x)). % reflexivity
```

As it turns out, this third formula is deducible from symmetry and transitivity taken together in the presence (or with the use of) condensed detachment. The last shortest single axiom that could ever be found was the following.

```
P(e(x,e(e(e(x,y),e(z,y)),z))). % XCB
```

This challenging formula was shown in 2002, in a collaboration of Ulrich and me, to be a single axiom for equivalential calculus. With that proof, all of the shortest single axioms have been found, all fourteen of length eleven. Repeated experiments relying on the subformula strategy to prove, with condensed detachment, that *XCB* is a single axiom met with no success, where the targets were the other thirteen single axioms and the cited 2-basis. I have no clue in the context of success versus failure when the subformula strategy is essentially the only strategy in use. Another mystery remains unsolved. Indeed, perhaps more generally, what property or properties strongly indicates that the subformula strategy will be quite effective?

Next are remarks and an experiment focusing on the use of demodulation to block one or more items with the goal of finding a proof shorter than that in hand. You have at the beginning of this section an example of success when a step of a 22-step proof was blocked from participating, with the result that a 21-step proof was found. The way I think of the situation is that a so-called unwanted step can, sometimes, be present in a proof that sort of forces the program to reason to a cul de sac, producing a proof longer than need be. By blocking the use of such a step, the program is free to follow a different—perhaps quite different—path that terminates with a more pleasing and shorter proof. Consistent with the spirit of this observation, I offer the following case, one that does in fact occur.

As I have noted in various notebooks, when seeking the proof of two or more items, sometimes a shorter proof than in hand of their join is found with the property that one or more of the subproofs (of members of the conjunction) is lengthened. More specifically, you might have a proof of, say, length 30 that contains, respectively, proofs (of the three members of the join) of length 18, 10, and 8. You might then find a proof of length 27 (of the conjunction) containing proofs of length, respectively, 18, 12, and 11. Further, the 18-step subproof within the 27-step proof is precisely that occurring in the 30-step proof. What has occurred is the (so-to-speak) forcing of steps of the 18-step proof into subproofs of the other two members, making such steps do double duty, triple duty, or more. You are again visiting the essence of the cramming strategy. I have had in the back of my mind—late in my list evidently—the experiment that chooses, say, a 5-step proof in place of a 4-step proof, with the goal of finding a shorter proof than that relying on the 4-step subproof. My delay in pursuing this idea for some time is explained in part by my study of other aspects of automated reasoning and proof search. I did finally conduct the experiment, however. The context was *MV*-implicational, the hypothesis the Rezus-style single axiom for that area, and the target the join of *MV*1, *MV*2, *MV*3, and *MV*5 (the 4-basis studied earlier in this notebook. In the first of two experiments, OTTER was permitted to find a 5-step proof of *MV*5, closely followed by a 4-step proof, which led to an 18-step proof of the join. In the second of the two experiments, with the inclusion of a single demodulator that blocked the third step of the 4-step proof, OTTER was prevented from finding the 4-step proof, which led to the completion of a 17-step proof. The cited step-blocking prevented the program from finding the shorter 4-step proof of *MV*5, forcing the program to build on a 5-step proof. In other words, I (in effect) preferred the use of a longer subproof over a shorter subproof, 5 versus 4 in length. If the following input file is used as given, you will be conducting the second experiment. If, instead, you comment out the demodulator and assign the value 6 (rather the 5) to max_proofs, you can conduct the first experiment.

An Input File Illustrating the Power of Demodulation Blocking

```
set(hyper_res).
assign(max_mem,880000).
% assign(max_seconds,3).
% set(sos_queue).
set(input_sos_first).
assign(max_weight,15).
% assign(change_limit_after,50).
% assign(new_max_weight,6).
assign(max_proofs,5).
% assign(max_distinct_vars,19).
assign(pick_given_ratio,4).
assign(bsub_hint_wt,1).
assign(report,5400).
set(keep_hint_subsumers).
% set(keep_hint_equivalents).
set(ancestor_subsume).
set(back_sub).
set(order_history).
```

```
% set(process_input).
clear(print_kept).

weight_list(pick_and_purge).
% Following 15 prove one of the 4 basis elements, with one step not in the 18,
% temp.ulrich.rezus.mv.out6a.
weight(P(i(x,i(y,i(z,y)))),0).
weight(P(i(x,i(y,x))),0).
weight(P(i(i(i(i(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8),
  i(i(i(i(v9,v10),i(v10,v9)),i(v10,v9)),v11)),v11)),0).
weight(P(i(i(i(i(i(x,y),
  i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w),i(i(i(v6,v7),v7),i(i(v7,v6),v6)),v8)),v8)),0).
weight(P(i(i(i(i(x,y),i(i(z,x),i(z,y))),i(i(u,i(v,u)),w)),w)),0).
weight(P(i(x,i(i(i(y,z),z),i(i(z,y),y)))),0).
weight(P(i(i(x,y),i(i(z,x),i(z,y)))),0).
weight(P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))),0).
weight(P(i(i(i(i(i(x,i(y,x)),i(x,z)),i(i(y,x),z)),i(i(y,x),z)),i(x,z))),0).
weight(P(i(i(x,i(i(i(y,z),i(i(u,y),i(u,z))),i(i(v,i(w,v)),v6))),i(x,v6))),0).
weight(P(i(i(x,i(y,z)),i(x,i(y,i(u,z))))),0).
weight(P(i(x,i(i(x,y),y))),0).
weight(P(i(x,i(i(y,i(x,z)),i(y,z)))),0).
weight(P(i(x,i(i(y,i(x,z)),i(u,i(y,z))))),0).
weight(P(i(i(x,y),i(i(y,z),i(x,z)))),0).
% Following 25, including target, are subformulas of the Rezus for the implicational of MV,
% from Ross.
weight(i(v,u),1).
weight(i(x,i(y,x)),1).
weight(i(w,i(v6,w)),1).
weight(i(i(v8,v9),v9),1).
weight(i(i(v9,v8),v8),1).
weight(i(i(v,z),i(v,u)),1).
weight(i(v14,i(v15,v14)),1).
weight(i(i(w,i(v6,w)),v7),1).
weight(i(i(v11,v12),i(v12,v11)),1).
weight(i(i(v14,i(v15,v14)),v16),1).
weight(i(i(z,u),i(i(v,z),i(v,u))),1).
weight(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),1).
weight(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),1).
weight(i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10),1).
weight(i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13),1).
weight(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),1).
weight(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),1).
weight(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),i(i(v9,v8),v8)),v10)),1).
weight(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),1).
weight(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),1).
weight(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),1).
weight(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16)),1).
weight(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(i(v8,v9),v9),
  i(i(v9,v8),v8)),v10)),v10),i(i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),i(i(v14,i(v15,v14)),v16))),1).
```

```
weight(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(v8,v9),v9),
  i(i(v9,v8),v8))),v10)),v10),i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
    i(i(v14,i(v15,v14)),v16))),v16),1).
weight(P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),
  i(i(i(v8,v9),v9),i(i(v9,v8),v8))),v10)),v10),i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  i(i(v14,i(v15,v14)),v16))),v16)),1).  % Rezus-style axiom for MV
end_of_list.

list(usable).
-P(i(x,y)) | -P(x) | P(y).
-P(i(a,i(b,a))) | -P(i(i(a,b),i(i(b,c),i(a,c)))) | -P(i(i(i(a,b),b),i(i(b,a),a))) | -P(i(i(i(a,b),i(b,a)),i(b,a))) |
  $ANS(MVBASISALL).
end_of_list.

list(sos).
P(i(i(i(x,i(y,x)),i(i(i(i(i(i(i(i(i(z,u),i(i(v,z),i(v,u))),i(i(w,i(v6,w)),v7)),v7),i(i(i(v8,v9),v9),
  i(i(v9,v8),v8))),v10)),v10),i(i(i(v11,v12),i(v12,v11)),i(v12,v11)),v13)),v13),
  (i(v14,i(v15,v14)),v16))),v16)).
% Rezus-style axiom for MV
end_of_list.

list(passive).
%  Following 4-basis, used by Veroff, for MV implicational fragment.
-P(i(a,i(b,a))) | $ANS(MVBASIS).
-P(i(i(a,b),i(i(b,c),i(a,c)))) | $ANS(MVBASIS).
-P(i(i(i(a,b),b),i(i(b,a),a))) | $ANS(MVBASIS).
-P(i(i(i(a,b),i(b,a)),i(b,a))) | $ANS(MVBASIS).
end_of_list.

list(demodulators).
(P(i(x,i(i(i(y,z),i(z,y)),i(z,y)))) = junk).
%  (i(i(x,x),y) = junk).
%  (i(y,i(x,x)) = junk).
(i(x,junk) = junk).
(i(junk,x) = junk).
(P(junk) = $T).
end_of_list.
```

By conducting both experiments you can witness apparent progress being made (in the first) as a shorter subproof is found of *MV*5, length 5 then 4, but in actuality a cul de sac is encountered, and a longer total proof is completed (18 versus 17).

At this point, I turn to answering various questions that might have occurred to you, some relevant to topics closely related to the earlier material in this notebook, and some only distantly related. For example, in this section, once I focused on avoiding the use of level saturation, avoided set(sos_queue), and once I relied upon it. So far in this notebook, three times I used level saturation, and seven times I did not, preferring a complexity-preference approach. In this context, you might more than reasonably ask about the advantages and disadvantages of either approach, breadth-first or complexity-preference. The use of a breadth-first search forces the program to consider for inference-rule initiation complex items, those that might otherwise never be chosen for this purpose. After all, certainly for OTTER, the choice of where next to focus the reasoning is based on simplest over less simple. But, sometimes, a quite complex formula or equation is needed somewhere in the sought-after proof. In such instances—although many, many years ago I did not recognize the need—a breadth-first search offers a far greater chance of success than does a complexity-preference approach. However, in that the levels often grow exponentially in size, usually level saturation will fail. Indeed, such a failure is preceded by the expenditure of much CPU time which, although free, tries one's patience.

You might be curious, as Beeson were, about the effectiveness of the subformula strategy when more than two connective (operators) are present. For some evidence in that context, you can study (as I did) intuitionistic propositional calculus, whose fragment was the focus in the second anecdote offered earlier in this section. L. E. J. Brouwer's student A. Heyting was the first, in 1930, to present an axiom system for the intuitionistic propositional calculus, a system designed to allow only those theorems and rules that intuitionists find acceptable in proving (constructively) results in mathematics. Where the function *i* denotes implication, *n* negation, *k* logical **and**, and *a* logical **or**, the following ten axioms, taken from Chapter 4 of a book I wrote titled "Automated Reasoning and the Discovery of Missing and Elegant Proofs", offer a slight simplification of Heyting's eleven axiom system.

```
P(i(x,i(y,x))).
P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
P(i(n(x),i(x,y))).
P(i(i(x,n(x)),n(x))).
P(i(k(x,y),x)).
P(i(k(x,y),y)).
P(i(i(x,y),i(i(x,z),i(x,k(y,z))))).
P(i(x,a(x,y))).
P(i(x,a(y,x))).
P(i(i(x,y),i(i(z,y),i(a(x,z),y)))).
```

With this set of ten axioms and the following input file, more evidence is provided for the power of the subformula strategy. In particular, the use of the input file eventually results in proofs of a number of interesting theorems that I will discuss.

An Input File for the Study of Intuitionistic Logic

```
% Otter file, by Beeson
% Intuitionistic propositional logic with axioms from Heyting's book.
% Goal to derive a different axiomatization from the Appendix of Prior's Formal Logic.

set(hyper_res).
assign(max_weight,9).
% assign(change_limit_after,200).
% assign(new_max_weight,6).
clear(print_kept).
set(process_input).
% set(ancestor_subsume).
set(back_sub).
clear(print_back_sub).
assign(max_distinct_vars,4).
assign(pick_given_ratio,2).
assign(max_mem,800000).
% assign(max_seconds,1).
% assign(max_given,24).
assign(max_proofs,-1).
assign(report,5400).
set(order_history).
set(input_sos_first).
% set(sos_queue).
set(keep_hint_subsumers).
```

```
assign(bsub_hint_wt,1).

weight_list(pick_and_purge).
% Following 81 sorted to remove duplicates are subformulas of hypotheses and targets.
weight(P(i(a(x,y),a(y,x))),1).
weight(P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))),1).
weight(P(i(i(x,i(y,z)),i(k(x,y),z))),1).
weight(P(i(i(x,i(y,z)),i(y,i(x,z)))),1).
weight(P(i(i(x,n(x)),n(x))),1).
weight(P(i(i(x,y),i(i(x,n(y)),n(y)))),1).
weight(P(i(i(x,y),i(i(x,z),i(x,k(y,z))))),1).
weight(P(i(i(x,y),i(i(z,y),i(a(x,z),y)))),1).
weight(P(i(i(x,y),i(k(x,z),k(y,z)))),1).
weight(P(i(i(x,z),i(i(y,z),i(a(x,y),z)))),1).
weight(P(i(i(y,z),i(i(x,y),i(x,z)))),1).
weight(P(i(k(i(x,y),i(x,n(y))),n(x))),1).
weight(P(i(k(i(x,y),i(y,z)),i(x,z))),1).
weight(P(i(k(i(x,z),i(y,z)),i(a(x,y),z))),1).
weight(P(i(k(x,i(x,y)),y)),1).
weight(P(i(k(x,y),k(y,x))),1).
weight(P(i(k(x,y),x)),1).
weight(P(i(k(x,y),y)),1).
weight(P(i(n(x),i(x,y))),1).
weight(P(i(x,a(x,y))),1).
weight(P(i(x,a(y,x))),1).
weight(P(i(x,i(n(x),y))),1).
weight(P(i(x,i(y,x))),1).
weight(P(i(x,k(x,x))),1).
weight(P(i(y,a(x,y))),1).
weight(P(i(y,i(x,y))),1).
weight(a(x,y),1).
weight(a(x,z),1).
weight(a(y,x),1).
weight(i(a(x,y),a(y,x)),1).
weight(i(a(x,y),z),1).
weight(i(a(x,z),y),1).
weight(i(i(x,i(y,z)),i(i(x,y),i(x,z))),1).
weight(i(i(x,i(y,z)),i(k(x,y),z)),1).
weight(i(i(x,i(y,z)),i(y,i(x,z))),1).
weight(i(i(x,n(x)),n(x)),1).
weight(i(i(x,n(y)),n(y)),1).
weight(i(i(x,y),i(i(x,n(y)),n(y))),1).
weight(i(i(x,y),i(i(x,z),i(x,k(y,z)))),1).
weight(i(i(x,y),i(i(z,y),i(a(x,z),y))),1).
weight(i(i(x,y),i(k(x,z),k(y,z))),1).
weight(i(i(x,y),i(x,z)),1).
weight(i(i(x,z),i(i(y,z),i(a(x,y),z))),1).
weight(i(i(x,z),i(x,k(y,z))),1).
weight(i(i(y,z),i(a(x,y),z)),1).
weight(i(i(y,z),i(i(x,y),i(x,z))),1).
weight(i(i(z,y),i(a(x,z),y)),1).
weight(i(k(i(x,y),i(x,n(y))),n(x)),1).
weight(i(k(i(x,y),i(y,z)),i(x,z)),1).
```

```
weight(i(k(i(x,z),i(y,z)),i(a(x,y),z)),1).
weight(i(k(x,i(x,y)),y),1).
weight(i(k(x,y),k(y,x)),1).
weight(i(k(x,y),x),1).
weight(i(k(x,y),y),1).
weight(i(k(x,y),z),1).
weight(i(k(x,z),k(y,z)),1).
weight(i(n(x),i(x,y)),1).
weight(i(n(x),y),1).
weight(i(x,a(x,y)),1).
weight(i(x,a(y,x)),1).
weight(i(x,i(n(x),y)),1).
weight(i(x,i(y,x)),1).
weight(i(x,i(y,z)),1).
weight(i(x,k(x,x)),1).
weight(i(x,k(y,z)),1).
weight(i(x,n(x)),1).
weight(i(x,n(y)),1).
weight(i(x,y),1).
weight(i(x,z),1).
weight(i(y,a(x,y)),1).
weight(i(y,i(x,y)),1).
weight(i(y,i(x,z)),1).
weight(i(y,x),1).
weight(i(y,z),1).
weight(i(z,y),1).
weight(k(i(x,y),i(x,n(y))),1).
weight(k(i(x,y),i(y,z)),1).
weight(k(i(x,z),i(y,z)),1).
weight(k(x,i(x,y)),1).
weight(k(x,x),1).
weight(n(x),1).
weight(junk,1000).
end_of_list.

list(usable).
% condensed detachment
-P(i(x,y)) | -P(x) | P(y).
end_of_list.

list(sos).
% % Following 10 is an axiomatization of intuitionistic logic from Horn's 1962 paper
% P(i(x,i(y,x))).    % 1
% P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).   %2
% P(i(k(x,y),x)).    % 3
% P(i(k(x,y),y)).    % 4
% P(i(i(x,y),i(i(x,z),i(x,k(y,z))))).   %5
% P(i(x,a(x,y))).         %6
% P(i(x,a(y,x))).         %7
% P(i(i(x,y),i(i(z,y),i(a(x,z),y)))). %8
% P(i(i(x,n(y)),i(y,n(x)))).   %9
% P(i(n(x),i(x,y))).      %10
% Following 10 is an axiomatization (10) of intuitionistic logic from my book4,
```

```
% simplified of Heyting's
P(i(x,i(y,x))).
P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
P(i(n(x),i(x,y))).
P(i(i(x,n(x)),n(x))).
P(i(k(x,y),x)).
P(i(k(x,y),y)).
P(i(i(x,y),i(i(x,z),i(x,k(y,z))))).
P(i(x,a(x,y))).
P(i(x,a(y,x))).
P(i(i(x,y),i(i(z,y),i(a(x,z),y)))).
end_of_list.

list(passive).
% Following 10 is an axiomatization of intuitionistic logic from my book4,
% simplified of Heyting's
-P(i(a1,i(a2,a1))) | $ANS(M01).
-P(i(i(a1,i(a2,a3)),i(i(a1,a2),i(a1,a3)))) | $ANS(M02).
-P(i(n(a1),i(a1,a2))) | $ANS(M03).
-P(i(i(a1,n(a1)),n(a1))) | $ANS(M04).
-P(i(k(a1,a2),a1)) | $ANS(M05).
-P(i(k(a1,a2),a2)) | $ANS(M06).
-P(i(i(a1,a2),i(i(a1,a3),i(a1,k(a2,a3))))) | $ANS(M07).
-P(i(a1,a(a1,a2))) | $ANS(M08).
-P(i(a1,a(a2,a1))) | $ANS(M09).
-P(i(i(a1,a2),i(i(a3,a2),i(a(a1,a3),a2)))) | $ANS(M10).
% Following 11 purport to be Heyting's from his book axioms.
-P(i(b,k(b,b))) | $ANS(01).
-P(i(k(b,c),k(c,b))) | $ANS(02).
-P(i(i(b,c),i(k(b,d),k(c,d)))) | $ANS(03).
-P(i(k(i(b,c),i(c,d)),i(b,d))) | $ANS(04).
-P(i(c,i(b,c))) | $ANS(05).
-P(i(k(b,i(b,c)),c)) | $ANS(06).
-P(i(b,a(b,c))) | $ANS(07).
-P(i(a(b,c),a(c,b))) | $ANS(08).
-P(i(k(i(b,d),i(c,d)),i(a(b,c),d))) | $ANS(09).
-P(i(n(b),i(b,c))) | $ANS(10).
-P(i(k(i(b,c),i(b,n(c))),n(b))) | $ANS(11).
% Following 2 aux theorems from Ulrich.
-P(i(i(p,q),i(i(p,n(q)),n(p)))) | $ANS(11A).
-P(i(i(p,i(q,r)),i(k(p,q),r))) | $ANS(11AA).
% Following 11 theorems to prove.
-P(i(i(q,r),i(i(p,q),i(p,r))))  | $ANS(H2).
-P(i(i(p,i(q,r)),i(q,i(p,r))))  | $ANS(H3).
-P(i(p,i(q,p)))                 | $ANS(H4).
-P(i(k(p,q),p))                 | $ANS(K1).
-P(i(k(p,q),q))                 | $ANS(K2).
-P(i(i(p,q),i(i(p,r),i(p,k(q,r))))) | $ANS(K3).
-P(i(p,a(p,q)))                 | $ANS(A1).
-P(i(q,a(p,q)))                 | $ANS(A2).
-P(i(i(p,r),i(i(q,r),i(a(p,q),r)))) | $ANS(A3).
-P(i(i(p,n(p)),n(p)))           | $ANS(N1).
-P(i(p,i(n(p),q))) | $ANS(N2).
```

```
% Following 7 theorems from Heyting's paper.
-P(i(k(p,q),p)) | $ANS(B22).
-P(i(p,p)) | $ANS(B221).
-P(i(k(p,q),q)) | $ANS(B222).
-P(i(k(i(p,q),i(r,d)),i(k(p,r),k(q,d)))) | $ANS(B223).
-P(iff(k(i(p,q),i(p,r)),i(p,k(q,r)))) | $ANS(B224).
-P(i(k(p,i(q,r)),i(q,k(p,r)))) | $ANS(B225).
-P(i(p,i(q,k(q,p)))) | $ANS(B226).
% Following is that axiom in the ten sent to me by Beeson not among the ten in my book.
-P(i(i(p,n(q)),i(q,n(p)))) | $ANS(HORN).
end_of_list.

list(demodulators).
% n(n(n(x))) = junk.
n(n(x)) = junk.
i(x,junk) = junk.
i(junk,x) = junk.
n(junk) = junk.
end_of_list.
```

You find in the input file just given weight templates for 81 subformulas. The large number results from choosing subformulas from the ten axioms for intuitionism and from various theorems to be proved. I included the command set(process_input) to enable me to see the proofs of length 0 of the ten hypotheses as well as those, also of length 0, items that are simultaneously among the ten and among the theorems to prove. Yes, an overlap does exist. As for the theorems to prove, whose negations are found is list(passive), eleven are an axiom system offered by Heyting for this area of logic. Two of the theorems were suggested by Ulrich. The next eleven are theorems suggested by Beeson, I suspect taken from the work of Heyting. The next seven are indeed theorems from Heyting, again supplied by Beeson. The last theorem to prove corresponds to an axiom, among ten, offered by Horn for this area of logic. A word now is in order concerning two other axiom systems that differ from the ten-axiom system found in list(sos).

Horn offered a ten-axiom system that is nearly identical to the ten I used. The difference is that the fourth in the set I used, the first of the following two, is replaced by the second of the two that follows.

```
P(i(i(x,n(x)),n(x))).
P(i(i(x,n(y)),i(y,n(x)))).
```

If you wish to experiment with the Horn system, which I have, you merely take the given input file, remove the comment from the first eleven items in list(sos), and insert in column 1 a comment (%) of the next eleven. On the other hand, if you wish to experiment with the eleven-axiom system of Heyting, you replace in the given input file all that resides there by the following eleven clauses.

```
P(i(x,k(x,x))).
P(i(k(x,y),k(y,x))).
P(i(i(x,y),i(k(x,z),k(y,z)))).
P(i(k(i(x,y),i(y,z)),i(x,z))).
P(i(y,i(x,y))).
P(i(k(x,i(x,y)),y)).
P(i(x,a(x,y))).
P(i(a(x,y),a(y,x))).
P(i(k(i(x,z),i(y,z)),i(a(x,y),z))).
P(i(n(x),i(x,y))).
P(i(k(i(x,y),i(x,n(y))),n(x))).
```

When I made this replacement, relying on the eleven Heyting axioms, I was unable to prove many of the theorems whose negation is found in list(passive) of the preceding input file; further, I was not able to deduce, from the Heyting eleven-axiom system, the ten-axiom system I used. My initial suspicion was that this set of eleven does *not* axiomatize intuitionism. Later, I learned that nothing was wrong with the eleven-axiom system; what was needed was, in addition to condensed detachment, another inference rule.

On the other hand, the use of the file with the replacement cited did result in the completion of proofs of all so-called eleven Heyting axioms. The use of the input file also yielded proofs of all (items in the passive list) but B223 and B224 (suggested by Beeson), whose proofs I leave to you. The most difficult theorem to prove was that of deducing the axiom in the Horn set that is not in the set I used (designated by HORN in the input file), a proof of length 44, level 12, and requiring nearly 3 CPU-hours to complete. All in all, I think it safe to say that the subformula strategy coped well with 81 subformulas and four different operators. However, you might not agree totally in that so much CPU time was required. (For the curious, when I experimented with a variant of the given input file in which level saturation was employed rather than pick_given_ratio, far less success resulted.) Quite a while later, after writing much of the material that follows, I sought and obtained much easier a proof of Horn.

therefore, the corresponding expperiment to try focuses on the use of the input file, but with the last six items in list(sos) commented out (with %). After all, the target, HORN, involves just the functions *i* and *n*, and only the first four axioms have that property. The relevant property is called *separable*, which means, if present, that the only axioms needed are those that precisely share the functions of the target. The experiment was most rewarding, returning a 24-step proof of level 11 in approximately 42 CPU-seconds, with retention of clause (21324). Clearly, as Beeson observed in an e-mail, with but two connectives, far fewer theorems are deduced, and the search space is much smaller.

One more item remains that focuses on intuitionism, actually, on the implicational fragment studied earlier in this section. Theorems of that area of logic are theorems of the far-more-familiar classical propositional calculus. Therefore, the twelve single axiom—the ten from Ulrich and the two from Meredith—are theorems of propositional logic and, in principle, provable with condensed detachment. I did in fact run the corresponding experiment, using as hypothesis the following three-axiom system from Lukasiewicz.

```
% Luka 1 2 3.
P(i(i(x,y),i(i(y,z),i(x,z)))).
P(i(i(n(x),x),x)).
P(i(x,i(n(x),y))).
```

With 68 subformulas directing the program's reasoning, OTTER completed proofs of ten of the twelve single axioms for positive implication. The two that were not proved are Ulrich's sixth and seventh, which I leave to you for the moment, noting that later in this section I revisit this aspect through a different experiment. The last proved was Ulrich's fourth single axiom, with a length of 141, level of 47, and requiring approximately 2.5 CPU-hours. Now, for a fair amount of time, I was puzzled, wondering why the proof took so long. Do you see why, when compared with using the Hilbert 4-basis?

```
% Hilbert's 4-base for positive implicational logic (1922)
P(i(i(x,i(x,y)),i(x,y))).      %H1
P(i(i(y,z),i(i(x,y),i(x,z)))). %H2
P(i(i(x,i(y,z)),i(y,i(x,z)))). %H3
P(i(x,i(y,x))).                %H4
```

While you ponder, let me ask another question. Do you have a conjecture that explains why the sixth and seventh Ulrich single axioms are apparently harder to prove than his other eight—actually, were not proved—in the experiment?

Now, assuming enough time has elapsed to permit you to hazard a theory, I now explain my view of why so much CPU time was required.

Keeping in view the just-cited Hilbert 4-basis and the three-axiom system of Lukasiewicz, I re-examined the following.

```
% Following 10 negations of Ulrich single axioms.
-P(i(i(a4,i(a5,a6)),i(a1,i(i(i(a2,a4),a5),i(a4,a6))))) | $ANS(UL01).  % HI-3
-P(i(a4,i(i(a5,i(a6,a1)),i(i(i(a2,a5),a6),i(a5,a1))))) | $ANS(UL02).  % HI-4
```

```
-P(i(i(i(a4,a5),a6),i(i(a5,i(a6,a1)),i(a2,i(a5,a1))))) | $ANS(UL03).  % HI-5
-P(i(i(a4,i(a5,a6)),i(i(i(a1,a4),a5),i(a2,i(a4,a6))))) | $ANS(UL04).  % HI-6
-P(i(a4,i(i(i(a5,a6),a1),i(i(a6,i(a1,a2)),i(a6,a2))))) | $ANS(UL05).  % HI-7
-P(i(i(a4,a5),i(i(i(a6,i(a1,a4)),i(a5,a2)),i(a4,a2)))) | $ANS(UL06).  % HI-8
-P(i(i(a4,a5),i(i(a5,i(i(a6,a4),a1)),i(a2,i(a4,a1))))) | $ANS(UL07).  % HI-9
-P(i(i(a4,a5),i(a6,i(i(i(a1,a4),i(a5,a2)),i(a4,a2))))) | $ANS(UL08).  % HI-10
-P(i(a4,i(i(i(a5,a6),a1),i(i(a1,i(a1,a2)),i(a6,a2))))) | $ANS(UL09).  % HI-11
-P(i(i(i(a4,a5),a6),i(a1,i(i(a6,i(a6,a2)),i(a5,a2))))) | $ANS(UL10).  % HI-12
%  Meredith's two single axioms
-P(i(i(i(p,q),r),i(s,i(i(q,i(r,t)),i(q,t))))) | $ANS(Meredith_1A).
-P(i(t,i(i(p,q),i(i(i(s,p),i(q,r)),i(p,r))))) | $ANS(Meredith_1B).
```

I noted that the only function present in the twelve targets is *i* for implication. The only function present in the Hilbert 4-basis is also *i*. In the Lukasiewicz three-axiom system, however, two functions are present, *i* and *n*, the latter for negation. Further, only one of the three Lukasiewicz axioms relies on *i* alone. Therefore, I concluded that although his second and third axioms come into play, both relying in part on *n*, the presence of that function (in effect) gets in the way. I think this explanation is more than plausible.

If you should decide to seek a shorter proof for one or more of these theorems, for example, a shorter proof of the member of the ten-axiom Horn system that is not a member of the ten-axiom system I used to study intuitionism, the cramming strategy offers an interesting approach. Indeed, in addition to when the target is a conjunction of two or more items, cramming has proved in various cases to be successful. What you do is first negate all of the proof steps of the proof you have in hand, with the objective of obtaining subproofs of the total proof of the theorem under study. Those negations are placed in list(passive). You could include the command set(ancestor_subsume), instructing the program to seek shorter proofs on its own. You then choose one of the subproofs and place all of the deduced steps in a new input file, placing them in list(sos) and relying on set(input_sos_first). That last command forces a reasoning program to focus, for inference-rule initiation, on the elements of the initial set of support before focusing on any deduced clause. The intention is to force—cram—those added steps into the final proof (of the target) in a manner that they play double duty, triple duty, or more, with the result that a shorter proof is completed. The idea is to replace steps in the so-called first proof by steps already present and used in other deductions of steps of a proof. You are thus cramming on a subproof (within the total proof) of an intermediate step of a proof in hand with the intention of discovering a shorter proof. Summarizing, in the spirit of cramming where the target is a conjunction, the just-described approach (concerned with proving a single item) asks you to select an intermediate step *B* of the proof of *A* and seek a shorter proof of the (in effect) conjunction of *B* and *A*. By the way, cramming has proved useful in various aspects of algebra (where equations dominate the study), both when the target is a conjunction and when it is a single element.

Earlier, I said I would revisit the attempt to prove all twelve single axioms for the implicational fragment of intuitionism when the hypothesis consists of the Lukasiewicz three-axiom system. At that point, I asked you to explain why the sixth and seventh single axioms of Ulrich were so hard to prove, actually were not proved. Well, I found a way to prove all twelve in one run, a run that still relies on just the 68 subformulas. I made two changes to my previous approach. First, I changed max_weight, by assigning the value 12 instead of 32. Second, with the inclusion of the following, I instructed OTTER to purge upon generation any new conclusion that relies on double negation.

```
list(demodulators).
%  n(n(n(x))) = junk.
n(n(x)) = junk.
i(x,junk) = junk.
i(junk,x) = junk.
n(junk) = junk.
end_of_list.
```

In contrast to the run relying on an assignment of 32 to max_weight and containing no demodulators, which ran out of memory, my second attempt (on a much faster computer) proved, after the Ulrich fourth single axiom, both his seventh and sixth, in that order. Those last two proofs required approximately ten CPU-hours to complete. I cannot even guess how important to the completion of proofs of all twelve was the inclusion of the given demodulators, for the first attempt contained only 36 double-negation formulas. However, in numerous other experiments spread over many, many years, blocking double-negation items has proved to be a powerful move, often leading to the discovery of an unexpectedly short proof.

I return to a perhaps surprising observation about using subformulas. I ran an experiment a couple of weeks ago in which the goal was to prove, from the Meredith single axiom for classical propositional calculus, the Lukasiewicz three-axiom system, that consisting of 1, 2, and 3 (of his 71 theses). Inadvertently, I relied on the wrong set of subformulas, subformulas not taken from the Meredith axiom or from the three targets. I—actually, OTTER—succeeded, suggesting that even a possibly poor choice of subformulas can sometimes lead to reaching the objective. For a somewhat related comment, it appears that having more subformulas can result in faster completion of the sought-after proof(s). At least, that was the case when I studied the Meredith 21-letter single axiom (for propositional calculus) versus the Lukasiewicz 23-letter single axiom for that area of logic; the latter offers a great number of subformulas. In both studies, I blocked the retention of double-negation formulas, which is so typical of my experimentation. Indeed, from a far from thorough exploration, I feel certain that such blocking is advantageous when measured in terms of the CPU time required to complete the task at hand.

Two experiments focusing on the following Meredith single axiom (for classical propositional calculus) and diverse uses of the subformula strategy are, to me, interesting.

```
% Following is Meredith's axiom.
P(i(i(i(i(i(x,y),i(n(z),n(u))),z),v),i(i(v,x),i(u,x)))).
```

The only difference, which in fact turned out to have a significant effect, was the inclusion in the second experiment of subformulas of members of known axiom system, 57 additional subformulas. Before, reading the text that follows the input file I now give, you might enjoy conjecturing about the effects of including the additional 57 subformulas.

An Input File Using the Meredith Single Axiom to Deduce Another Axiom system

```
% Trying for an automated proof of Meredith's single axiom with the double-negation property.
set(hyper_res).
assign(max_weight,12).
% assign(change_limit_after,200).
% assign(new_max_weight,20).
assign(max_proofs,-1).
clear(print_kept).
% set(process_input).
% set(ancestor_subsume).
set(back_sub).
clear(print_back_sub).
assign(max_distinct_vars,7).
assign(pick_given_ratio,2).
assign(max_mem,880000).
% assign(max_seconds,120).
assign(report,3600).
set(order_history).
set(input_sos_first).
% set(sos_queue).
assign(heat,0).
% assign(dynamic_heat_weight,0).

weight_list(pick_and_purge).
```

```
% Following 3 are some of the subformulas of Luka123.
weight(i(x,y),1).
weight(i(i(y,z),i(x,z)),1).
weight(i(n(x),x),1).
% Following are Luka123 themselves.
weight(P(i(i(x,y),i(i(y,z),i(x,z)))),1).
weight(P(i(i(n(x),x),x)),1).
weight(P(i(x,i(n(x),y))),1).
% Following 8 are subformulas of Meredith single.
weight(i(u,x),1).
weight(i(n(z),n(u)),1).
weight(i(i(v,x),i(u,x)),1).
weight(i(i(x,y),i(n(z),n(u))),1).
weight(i(i(i(x,y),i(n(z),n(u))),z),1).
weight(i(i(i(i(x,y),i(n(z),n(u))),z),v),1).
weight(i(i(i(i(i(x,y),i(n(z),n(u))),z),v),i(i(v,x),i(u,x))),1).
weight(P(i(i(i(i(i(x,y),i(n(z),n(u))),z),v),i(i(v,x),i(u,x)))),1).
% Following 57 sorted subformulas of 17 members of known axiom systems.
weight(i(i(i(x,y),z),i(n(x),z)),2).
weight(i(i(i(x,y),z),i(y,z)),2).
weight(i(i(n(x),n(y)),i(y,x)),2).
weight(i(i(n(x),x),x),2).
weight(i(i(n(x),y),y),2).
weight(i(i(n(x),z),i(i(y,z),i(i(x,y),z))),2).
weight(i(i(u,i(n(x),z)),i(u,i(i(y,z),i(i(x,y),z)))),2).
weight(i(i(x,i(x,y)),i(x,y)),2).
weight(i(i(x,i(y,z)),i(i(x,y),i(x,z))),2).
weight(i(i(x,i(y,z)),i(y,i(x,z))),2).
weight(i(i(x,y),i(i(n(x),y),y)),2).
weight(i(i(x,y),i(i(y,w),i(x,w))),2).
weight(i(i(x,y),i(n(y),n(x))),2).
weight(i(i(x,y),i(x,z)),2).
weight(i(i(x,y),z),2).
weight(i(i(y,w),i(x,w)),2).
weight(i(i(y,z),i(i(x,y),i(x,z))),2).
weight(i(i(y,z),i(i(x,y),z)),2).
weight(i(n(n(x)),x),2).
weight(i(n(x),n(y)),2).
weight(i(n(x),x),2).
weight(i(n(x),y),2).
weight(i(n(x),z),2).
weight(i(n(y),n(x)),2).
weight(i(u,i(i(y,z),i(i(x,y),z))),2).
weight(i(u,i(n(x),z)),2).
weight(i(x,i(n(x),y)),2).
weight(i(x,i(x,y)),2).
weight(i(x,i(y,z)),2).
weight(i(x,n(n(x))),2).
weight(i(x,y),2).
weight(i(x,z),2).
weight(i(y,i(x,y)),2).
weight(i(y,i(x,z)),2).
weight(i(y,w),2).
```

```
weight(i(y,x),2).
weight(i(y,z),2).
weight(n(n(x)),2).
weight(n(x),2).
weight(n(y),2).
weight(P(i(i(i(x,y),z),i(n(x),z))),2).
weight(P(i(i(i(x,y),z),i(y,z))),2).
weight(P(i(i(n(x),n(y)),i(y,x))),2).
weight(P(i(i(n(x),x),x)),2).
weight(P(i(i(n(x),z),i(i(y,z),i(i(x,y),z)))),2).
weight(P(i(i(u,i(n(x),z)),i(u,i(i(y,z),i(i(x,y),z))))),2).
weight(P(i(i(x,i(x,y)),i(x,y))),2).
weight(P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))),2).
weight(P(i(i(x,i(y,z)),i(y,i(x,z)))),2).
weight(P(i(i(x,y),i(i(n(x),y),y))),2).
weight(P(i(i(x,y),i(i(y,w),i(x,w)))),2).
weight(P(i(i(x,y),i(n(y),n(x)))),2).
weight(P(i(i(y,z),i(i(x,y),i(x,z)))),2).
weight(P(i(n(n(x)),x)),2).
weight(P(i(x,i(n(x),y))),2).
weight(P(i(x,n(n(x)))),2).
weight(P(i(y,i(x,y))),2).
end_of_list.

list(usable).
% condensed detachment
-P(i(x,y)) | -P(x) | P(y).
%  The following disjunctions are known axiom systems.
-P(i(q,i(p,q))) | -P(i(i(p,i(q,r)),i(i(p,q),i(p,r)))) | -P(i(n(n(p)),p)) | -P(i(p,n(n(p)))) |
  -P(i(i(p,q),i(n(q),n(p)))) | -P(i(i(p,i(q,r)),i(q,i(p,r)))) | $ANS(step_allFrege_18_35_39_40_46_21).
% 21 is dependent.
-P(i(q,i(p,q))) | -P(i(i(p,i(q,r)),i(q,i(p,r)))) | -P(i(i(q,r),i(i(p,q),i(p,r)))) | -P(i(p,i(n(p),q))) |
  -P(i(i(p,q),i(i(n(p),q),q))) | -P(i(i(p,i(p,q)),i(p,q))) | $ANS(step_allHilbert_18_21_22_3_54_30).
% 30 is dependent.
-P(i(q,i(p,q))) | -P(i(i(p,i(q,r)),i(i(p,q),i(p,r)))) | -P(i(i(n(p),n(q)),i(q,p))) |
  $ANS(step_allBEH_Church_FL_18_35_49).
-P(i(i(i(p,q),r),i(q,r))) | -P(i(i(i(p,q),r),i(n(p),r))) | -P(i(i(n(p),r),i(i(q,r),i(i(p,q),r)))) |
  $ANS(step_allLuka_x_19_37_59).
-P(i(i(i(p,q),r),i(q,r))) | -P(i(i(i(p,q),r),i(n(p),r))) | -P(i(i(s,i(n(p),r)),i(s,i(i(q,r),i(i(p,q),r))))) |
  $ANS(step_allWos_x_19_37_60).
-P(i(i(p,q),i(i(q,r),i(p,r)))) | -P(i(i(n(p),p),p)) | -P(i(p,i(n(p),q))) | $ANS(step_allLuka_1_2_3).
end_of_list.

list(sos).
%  Following is Meredith's axiom.
P(i(i(i(i(i(x,y),i(n(z),n(u))),z),v),i(i(v,x),i(u,x)))).
end_of_list.

list(passive).
%  Following are members of known axiom systems for two-valued.
-P(i(i(p,q),i(i(q,r),i(p,r)))) | $ANS(step_L1).
-P(i(i(n(p),p),p)) | $ANS(step_L2).
-P(i(p,i(n(p),q))) | $ANS(step_L3).
```

```
-P(i(q,i(p,q))) | $ANS(step_18).
-P(i(i(i(p,q),r),i(q,r))) | $ANS(step_19).
-P(i(i(p,i(q,r)),i(q,i(p,r)))) | $ANS(step_21).
-P(i(i(q,r),i(i(p,q),i(p,r)))) | $ANS(step_22).
-P(i(i(p,i(p,q)),i(p,q))) | $ANS(step_30).
-P(i(i(p,i(q,r)),i(i(p,q),i(p,r)))) | $ANS(step_35).
-P(i(i(i(p,q),r),i(n(p),r))) | $ANS(step_37).
-P(i(n(n(p)),p)) | $ANS(step_39).
-P(i(p,n(n(p)))) | $ANS(step_40).
-P(i(i(p,q),i(n(q),n(p)))) | $ANS(step_46).
-P(i(i(n(p),n(q)),i(q,p))) | $ANS(step_49).
-P(i(i(p,q),i(i(n(p),q),q))) | $ANS(step_54).
-P(i(i(n(p),r),i(i(q,r),i(i(p,q),r)))) | $ANS(step_59).
-P(i(i(s,i(n(p),r)),i(s,i(i(q,r),i(i(p,q),r))))) | $ANS(step_60).
end_of_list.

list(demodulators).
% (n(n(n(x))) = junk).
(n(n(x)) = junk).
% (i(i(x,x),y) = junk).
% (i(y,i(x,x)) = junk).
% (i(n(i(x,x)),y) = junk).
% (i(y,n(i(x,x))) = junk).
(i(x,junk) = junk).
(i(junk,x) = junk).
(n(junk) = junk).
(P(junk) = $T).
end_of_list.

list(hot).
-P(i(x,y)) | -P(x) | P(y).
% Following is Meredith's axiom.
P(i(i(i(i(i(x,y),i(n(z),n(u))),z),v),i(i(v,x),i(u,x)))).
% % Following is MV1 -- MV4.
% P(i(x,i(y,x))).
% P(i(i(x,y),i(i(y,z),i(x,z)))).
% P(i(i(i(x,y),y),i(i(y,x),x))).
% P(i(i(n(x),n(y)),i(y,x))).
end_of_list.
```

This input file was used in the second experiment. Have you conjectured what happened when I conducted the first experiment, one with the fifty-eight lines that include the 57 added subformulas commented out?

The following resulted. In the first experiment, a proof of the join of the three axioms of the Lukasiewicz system was obtained with retention of clause (215839) in approximately 9900 CPU-seconds, a proof of length 183 and level 54. In the second experiment, conducted about one day later, the same join was proved, but with retention of clause (90353) in approximately 1360 CPU-seconds, a proof of length 135 and level 41. My conclusion asserts that the added subformulas proved most useful. (You might find it interesting that a proof from the Meredith single axiom of the Lukasiewicz three-axiom system required more CPU time, if I remember correctly, and more effort on my part some years ago, before the formulation of the subformula strategy. The approach I used those years ago was iterative, relying on adjoining lemmas proved in one run to the input for a later run.)

As I did many years ago when I formulated the lemma-adjunction approach, I now (in effect) imitated what I had done before by switching my attention to the Lukasiewicz 23-letter single axiom for classical propositional calculus, the following.

```
% Following is Lukasiewicz's 23-letter single axiom.
P(i(i(i(x,y),i(i(i(n(z),n(u)),v),z)),i(w,i(i(z,x),i(u,x))))).
```

In particular, since the preceding experiment focusing on the Meredith single axiom was so satisfying, perhaps a comparable study of the Lukasiewicz axiom would be. I made the smallest changes to the preceding input file. Of course, I changed the list(sos) by inserting, in place of Meredith, the Lukasiewicz 23-letter formula. I also made the following insertions in place of their correspondents.

```
assign(max_distinct_vars,6).
% Following 10 are subformulas of Lukasiewicz 23-letter single axiom.
weight(n(z),2).
weight(i(x,y),2).
weight(i(n(z),n(u)),2).
weight(i(i(z,x),i(u,x)),2).
weight(i(i(n(z),n(u)),v),2).
weight(i(w,i(i(z,x),i(u,x))),2).
weight(i(i(i(n(z),n(u)),v),z),2).
weight(i(i(x,y),i(i(i(n(z),n(u)),v),z)),2).
weight(i(i(i(x,y),i(i(i(n(z),n(u)),v),z)),i(w,i(i(z,x),i(u,x)))),2).
weight(P(i(i(i(x,y),i(i(i(n(z),n(u)),v),z)),i(w,i(i(z,x),i(u,x))))),2).
```

As predictable, I hoped for quick success, especially when compared with those years ago when four runs were required (with lemma adjunction, in one afternoon). In imitation of the just-reported experiments, I conducted two. In the first, I did not include the fifty-seven subformulas taken from seventeen members of known axiom systems; in the second, I did include them. The first experiment was conducted on July 7, 2008, the second on July 8. The first experiment returned to me a proof of length 162 and level 59, completing with retention of clause (68388), requiring approximately 222 CPU-seconds. The second experiment produced a proof (again of the three-axiom system of Lukasiewicz) of length 94 and level 49, with retention of clause (26546), in approximately 59 CPU-seconds. The additional fifty-seven subformulas, as before, enabled OTTER to complete a desired proof in significantly less CPU time. Of course, you might wonder about a justification for including the extra subformulas. The reason rests with the fact that targets other than the Lukasiewicz three-axiom system were acceptable, other axiom systems (as you see in the preceding input file). In case you are curious, from what I know, no proof, using the 23-letter formula as hypothesis and deducing a known axiom system for classical propositional calculus, was offered by the literature until a few years ago when OTTER and I attacked the question.

Again I turn to a radically different topic, that concerned with proof shortening and cul de sacs and finding shorter subproofs of the members of the join to be proved. The example now in focus involves proving, from the Lukasiewicz three-axiom system that flickers throughout this notebook, the join of thesis 18, thesis 35, and thesis 49 (in focus early in this section). You saw with perhaps delight a pair of proofs of respective lengths 21 and 22, a nested pair, obtained with the first input file found in this section. If you used the input file in question and proceeded, as I did, by adding the command set(ancestor_subsume), you would find in the output a fine example of proofs of the members (theses 18, 35, and 49) getting shorter while the proof of their join gets longer for the first three so-called total proofs, and then varies. Indeed, the first proof (of the join) has length 36, followed by proofs of length 45, 55, 52, 48, 47, 51, and 47. Interspersed are proofs of thesis 49, of length in order 30, 29, 28, 21, and 20. Three proofs of thesis 35 are present among the set of proofs, of respective length 34, 33, and 28. As for thesis 18, three proofs are completed, of length 18, 15, and 14. You thus have yet another illustration of how apparent progress, when measured in terms of shorter and still shorter proofs of members of a conjunction, is just that, apparent; in reality, ground may be being lost in the context of finding a shorter proof of the entire conjunction. The fault lies neither with ancestor subsumption nor with the actions of a powerful automated reasoning program. Rather, the so-called blame goes to the introduction of one or more formulas that lead to a shorter proof of a member but whose use leads the program into a cul de sac. Put another way, the formula or formulas, avoided in order to shorten a proof of a member, may actually be very useful in finding a shorter proof of the join or conjunction. Such formulas may be used repeatedly in the shorter proof of the conjunction.

A somewhat similar observation in the context of attempting to shorten a proof in hand applies to the level of a proof. For example, if you were relying on a level-saturation approach, with OTTER set(sos_queue), and foound a level *k* and length *j* proof, further exploration might lead to finding a far shorter proof. As an illustration, consider the case in which you find a level-20 and length-40 proof. Further, assume that, not counting the final step, the last two steps of the proof are each of level 19, and the application of, say, condensed detachment yields the target, a step of level 20. For this case, there might exist a level-21 proof such that its first step has two level-0 (input) parents, its second has level 2 from the first step and an input item, and so on until its twentieth step has level 20 from the preceding level-19 parent and an input item, and the final (target) item has level 21 from the cited level-20 item and an input element. In this case, definitely not mythical in spirit, termination of the run when the first proof has been found can lead to the erroneous conclusion that not much more can be gained in the context of the discovery of a short proof. If you wish to impress some colleague by demonstrating how misleading things can be, you simply produce an example in which the proof in hand has an early step of, say, level 4 but length 7, and a later step of level 8 and length 15, and then talk about how such can be extended. For example, you could talk about a level-20 proof of length 80 that can be replaced by a level-21 proof of length 21.

Often, in fact, as proofs get shorter, their level gets higher, as shown in the following experiment of mine taken from a study of *MV*-full whose objective was to prove a Rezus-style single axiom for that area of logic. Five proofs were found, of lengths in order 34, 33, 23, 21, and 20; the respective levels are 13, 14, 14, 14, and 15. Of course, the given study is not overwhelming, but it does illustrate what can occur. Especially if you are not using a level-saturation approach, and if you permit the program to find many, many proofs, you will sometimes encounter what to me is piquant, namely, the proof lengths do not follow a pattern, increasing, decreasing, increasing, more increasing in length while the levels also do not follow a pattern.

8. Summary and Additional Pursuits

This notebook, I believe, offers the first proofs of various theorems in *MV*-implicational and in *MV*-full. It also offers, implicitly and explicitly, challenges and even research topics that might prove stimulating.

If you decide to conduct research or simply play with problems from mathematics or logic, and if you add to the team an automated reasoning program, you will see, rather quickly I suspect, why strategy is virtually necessary. Without it, the program can too easily get lost, wandering among a huge number of conclusions. The fault is not automated reasoning. Indeed, unaided, an individual can encounter the same, without effective direction and restriction. Whether you seek a first proof or a shorter proof, many strategies can be employed. You can, for example, limit the complexity of newly deduced items (with max_weight), limit the retention of new deductions based on the number of distinct variables on which they rest (max_distinct_vars), or discard a newly deduced item with the (nonstandard) use of demodulation. You can outline a conjectured proof and ask the program to complete it. You can use resonators and hints to direct the reasoning or, if you are indeed skillful, employ Veroff's sketches, a most powerful technique indeed. If you are knowledgeable about some area of mathematics or logic, you can bring that expertise to the attack. Indeed, with cramming, rather than guessing at which intermediate step of a proof in hand should be the focus of attention, you can choose wisely where the emphasis should be placed.

Clearly, OTTER offers the researcher, or the individual unfamiliar with automated reasoning, a wide variety of weapons for attacking a question or problem. The following somewhat disjointed remarks are intended to aid you in choosing a particular weapon to address one of the challenges or to conduct an experiment.

Perhaps you would enjoy experimenting with one of the bases presented in this notebook. For example, I recently returned to the Wozniakowska 3-basis; recall that I had found a 28-step proof (reported near the end of Section 6). My goal was to improve on this, in the context of proof length. After much experimentation, on June 3, 2008, I found a 26-step proof that deduces, from the Wozniakowska 3-basis, a Rezus-style single axiom for *MV*-implicational. The 26-step proof contains two formulas not in the 28-step proof. The breakthrough resulted from the use of resonators (corresponding to proof steps of proofs found along the way), cramming (on the 25th step of a 28-step proof), demodulation blocking (of steps one at a time), and iteration. I also used at various points the subformula strategy and ancestor subsumption. What I have not done—and you may wish to do so—is to prove, from each chosen basis, the other bases. I expect such an experiment would be quite challenging.

Perhaps, instead, you wish to find a proof that avoids the use of one or more lemmas. In such a case, demodulation can be used to block the use of the unwanted lemma or lemmas. I have successfully conducted such studies. I have also conducted studies designed to avoid some type of term, double-negation terms in particular. But, even if your goal is not term avoidance, a sharp increase in program efficiency (measured in CPU time) often results from blocking double-negation terms.

If you are seeking a first proof, an iterative approach that sometimes succeeds has you begin by placing in list(passive) various lemmas that might be relevant. You then make a run with the goal of proving at least some of them. In the next run, if some are proved, you place the proved lemmas in the new list(sos) and continue in this fashion. Or, instead, you can place in the new list(sos) all of the proof steps of the proved lemmas. In addition, you can place the proof steps as resonators in, say, weight_list(pick_and_purge). A so-called hybrid has you place proved lemmas in the new list(sos) and the proof steps of their proofs as resonators in the weight_list.

Of course, as anyone familiar with my views knows, I find most appealing the prospect of finding, with OTTER's assistance, a proof shorter than offered by the literature. When millions of new conclusions are retained before the goal is reached, I make an inference different from that which some make. Specifically, I do not decide that the approach is ill-chosen, or that the program is inadequate, or that the path that an unaided researcher takes is what is needed. Instead, I conjecture that the question in focus is a hard one to answer, or I seek to formulate a more effective approach, or both. Sometimes I find that a small change in the value assigned to various parameters wins the game. Yes, the meshing of the different options is an art.

I hope that you will join me in such artistic pursuits. You need not be an expert in logic, mathematics, computer science, or automated reasoning to accept one of the challenges offered in this notebook or any of the other notebooks found on my website. If you do succeed in obtaining a shorter proof than I offer, or if you triumph in some other way that you conjecture would interest me, I would enjoy hearing from you. As this notebook and others that are on my website strongly suggest, much curiosity, great excitement, and intense joy are the ingredients of my feast.